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On classical mechanical systems with non-linear constraints

Gláucio Terra [a,*], Marcelo H. Kobayashi [b,1]

[a] *Departamento de Matemática Aplicada, Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão, 1010, 05508-090 São Paulo, SP, Brazil*

[b] *Departamento de Engenharia Mecânica, Instituto Superior Técnico, Secção de Mecânica Aeroespacial, Av. Rovisco Pais 1, 1049-001 Lisbon, Portugal*

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Abstract

In the present work, we analyze classical mechanical systems with non-linear constraints in the velocities. We prove that the d'Alembert–Chetaev trajectories of a constrained mechanical system satisfy both Gauss' principle of least constraint and Hölder's principle. In the case of a free mechanics, they also satisfy Hertz's principle of least curvature if the constraint manifold is a cone. We show that the Gibbs–Maggi–Appell (GMA) vector field (i.e. the second-order vector field which defines the d'Alembert–Chetaev trajectories) conserves energy for any potential energy if, and only if, the constraint is homogeneous (i.e. if the Liouville vector field is tangent to the constraint manifold). We introduce the Jacobi–Carathéodory metric tensor and prove Jacobi–Carathéodory's theorem assuming that the constraint manifold is a cone. Finally, we present a version of Liouville's theorem on the conservation of volume for the flow of the GMA vector field.

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1. Introduction

The aim of this paper is to develop a geometric formulation of the dynamics of non-linearly constrained mechanical systems based on Newton's law.

* Corresponding author.

E-mail addresses: glaucio@ime.usp.br (G. Terra), marcelo@popsrv.ist.utl.pt (M.H. Kobayashi).

[1] Present address: Department of Mechanical Engineering, University of Hawaii-Manoa, 2540 Dole Street – Holmes Hall 302, Honolulu HI 96822.

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The constrained mechanical system is modeled by the following setup. We consider a smooth finite dimensional manifold M, called the *configuration space* of the mechanical system, and a smooth function $\mathrm{K} : \mathsf{TM} \to \mathbb{R}$, called the *kinetic energy*, which we assume to be a positive definite quadratic form on each fiber of the *velocity phase space* TM. By polarization of this quadratic form on each fiber of TM, we obtain a smooth metric tensor $\mathfrak{g}$ on M, endowed of which it becomes a Riemannian manifold. The constraint is given by a smooth embedded submanifold $\mathcal{C}$ of the tangent bundle $\tau_{\mathsf{M}} : \mathsf{TM} \to \mathsf{M}$, such that the restriction $\tau_{\mathsf{M}}|_{\mathcal{C}} : \mathcal{C} \to \mathsf{M}$ is a submersion. This is Marle's [35] definition of a "regular constraint"; other formulations of systems with non-linear constraints may be found in [5,7,14,15,24,27–29,35–38,51–54], among others. $\mathcal{C}$ is called the *constraint manifold* (or simply *constraint*). We say that the constraint is *linear* if $\mathcal{C}$ is a vector sub-bundle of TM. The linear constraint case is well known has an extensive literature ranging from classical texts such as [4,20,43,55] to papers using modern differential geometry [8,11,18,25,26,31], among others. A curve γ on M is a *motion* or *trajectory compatible with the constraint*, or *horizontal* with respect to the constraint, if it is differentiable and its velocity lies in $\mathcal{C}$ almost everywhere on its domain. The dynamics of the mechanical system is given by a smooth fiber bundle morphism (i.e. it is a smooth map and preserves fibers) $\mathcal{F} : \mathsf{TM} \to \mathsf{T}^*\mathsf{M}$, called the *external force*. We say that the external force $\mathcal{F}$ *derives from a potential* $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$ if it is of the form $v_q \in \mathsf{TM} \mapsto -d\mathrm{V}(q) \in \mathsf{T}^*\mathsf{M}$.

In the *unconstrained* case, i.e. if $\mathcal{C} = \mathsf{TM}$, we say that a curve γ on M is a *motion* or *trajectory of the mechanical system* $(\mathsf{M}, \mathrm{K}, \mathcal{F})$ if it is a solution of *Newton's equation* [41]:

$$\mathcal{F}(\dot\gamma) = \mu(\nabla_t \dot\gamma), \tag{1}$$

where ∇ is the Levi–Civita connection of the Riemannian manifold $(\mathsf{M}, \mathfrak{g})$, ∇_t the induced covariant derivative on fields along the curve γ and $\mu = \mathfrak{g}^\flat : \mathsf{TM} \to \mathsf{T}^*\mathsf{M}$ the Legendre transformation induced by the metric tensor. Using the notation $\mathfrak{g}^\sharp := (\mathfrak{g}^\flat)^{-1} : \mathsf{T}^*\mathsf{M} \to \mathsf{TM}$ and $\mathcal{F}^\sharp := \mathfrak{g}^\sharp \circ \mathcal{F}$ (which we also call *external force*), we obtain the following equivalent and more frequently used form of equation (1):

$$\mathcal{F}^\sharp(\dot\gamma) = \nabla_t \dot\gamma. \tag{2}$$

Taking vertical lifts on both members of the last equation, we obtain $(T\dot\gamma/\mathrm{d}t) - \mathrm{H}_{\dot\gamma}(\dot\gamma) = \lambda_{\dot\gamma}(\mathcal{F}^\sharp(\dot\gamma))$, showing that the solutions of (2) are the *base integral curves* (i.e. the projections on M of its integral curves) of the second-order vector field $X_{\mathcal{F}} \in \mathfrak{D}^1(\mathsf{TM})$ defined by, for all $v_q \in \mathsf{TM}$, $X_{\mathcal{F}}(v_q) = \mathsf{S}(v_q) + \lambda_{v_q}(\mathcal{F}^\sharp(v_q))$, where S is the geodesic spray of $(\mathsf{M}, \mathfrak{g})$. $X_{\mathcal{F}}$ is called the *Gibbs–Maggi–Appell vector field* (GMA) of $(\mathsf{M}, \mathrm{K}, \mathcal{F})$—this nomenclature was suggested by Fusco and Oliva [18] in the context of linearly constrained mechanical systems.

In the general case, we define a *motion* or *trajectory of the constrained mechanical system* $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$ as a curve γ on M which is compatible with the constraint $\mathcal{C}$ and which satisfies *Newton's equation with reaction term R*:

$$\nabla_t \dot\gamma = \mathcal{F}^\sharp(\dot\gamma) + R(\dot\gamma) \tag{3}$$

for some fiber preserving map $R : \mathcal{C} \to \mathsf{TM}$, called the *reaction force field*. We assume that R is an *admissible* reaction in the sense of Definition 6, what ensures the existence of a second-order vector field $X^R_{\mathcal{C}}$ on $\mathcal{C}$ (i.e. a vector field $X^R_{\mathcal{C}} : \mathcal{C} \to \mathsf{T}\mathcal{C}$ such that $\mathsf{T}\tau_{\mathsf{M}} \circ X^R_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}}$)

whose base integral curves are the solutions of (3). This vector field is obtained by taking vertical lifts on both members of Eq. (3).

We show in the present work that a convenient choice $R = R^A$ of the admissible reaction force mentioned above, through a rule which generalizes d'Alembert's principle for linearly constrained systems, leads to the so-called *d'Alembert–Chetaev mechanics*. This paper focuses on the study of some properties of the flow of the vector field $X_{\mathcal{C}}^{R^A}$ obtained by this choice of the reaction force, called the GMA *vector field* of the constrained mechanical system $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$.

Historically, to the best of our knowledge, the first example of a mechanical system with non-linear constraints in the velocities was proposed by Appell [3] (which later has risen some criticism, see [39]). Since then, the theory for constraints that are non-linear in the velocities has attracted the interest of both the mathematical and the physical communities. A concrete example of a class of non-linear constraints which has been studied to some extent is provided by the so-called *isokinetic dynamics* in which the kinetic energy is constrained to be constant. This example, first proposed by Hoover [23], finds many interesting applications in non-equilibrium statistical mechanics (see, for example [19,23,45,56]). Also, recently, Cushman et al. [13] realized a classical particle with spin as a rigid body constrained to have a fixed value of the norm of the angular momentum. In a broad sense, a non-linear constraint may be regarded as a control system—see the servomechanism Example 1(c). In this case, the resulting reaction field provided by d'Alembert–Chetaev's principle may be understood as a non-linear control law which minimizes the strength of the reaction field—see Example 2.

Nowadays, the field of non-linearly constrained mechanical systems remains an active area of research and, as far as we know, many fundamental results that hold for an unconstrained or linearly constrained mechanical system, such as Liouville's Theorem, had not yet been established for non-linear constraints.

The organization of the paper is the following: in Section 2, we set up basic definitions and notation, and we introduce a technique which will be used to enounce and prove the results in a coordinate-free manner.

In Section 3, we enounce and describe the main results of the paper.

In Section 3.1, we define the concept of *admissible reaction field* for a constrained mechanical system $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$ and its *d'Alembert–Chetaev trajectories*. These trajectories are the solutions of Newton's equations with reaction term (3) for a certain choice of the admissible reaction R that has remarkable properties. We also prove that the d'Alembert–Chetaev trajectories of $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$ satisfy Gauss' principle of least constraint—see Theorem 1. As a corollary of the latter we obtain the so-called Gibbs–Appell form of the equations [43].

In Section 3.1 we prove that if the external force $\mathcal{F}$ derives from a potential $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$, the d'Alembert–Chetaev trajectories also satisfy Hölder's principle—Theorem 2. If the constraint manifold is a cone, they also satisfy Hertz's principle of least curvature in the case of free mechanics—see Theorem 3. At this point we should mention that, except for some minor modifications which occur in the various formulations of systems with non-linear constraints, the d'Alembert–Chetaev trajectories and Hölder's principle are well known and consolidated in the literature—see [5,15,36,37], among others. However, the characterization of these trajectories through Gauss' principle of least constraint—interpreted here as a principle of "minimal reaction force"—and through Hertz's principle of least geodesic curvature seems to be an original contribution.

In Section 3.3 we deal with the conservation of energy property and conditions under which the GMA vector field is Hamiltonian with respect to some Poisson structure on $\mathcal{C}$.

In Section 3.4 we prove a version of Jacobi–Carathéodory's theorem for constrained mechanical systems, provided that the external force derives from a potential $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$ and that the constraint manifold is a cone.

Finally, in Section 3.5 we generalize Liouville's theorem on the conservation of volume for non-linearly constrained mechanical systems. We obtain, as by-products, the extension of some results by Sasaki [46,47] on the Riemannian metric on the tangent bundle.

In Section 4, we prove the main results.

2. Basic notations and definitions

In this section we set up the notation and basic definitions.

M will denote a smooth connected finite dimensional manifold; TM (resp. $\mathsf{T}^*\mathsf{M}$) denotes the tangent (resp. cotangent) bundle of M and $\tau_{\mathsf{M}} : \mathsf{TM} \to \mathsf{M}$, $\tau^*_{\mathsf{M}} : \mathsf{T}^*\mathsf{M} \to \mathsf{M}$ the associated projections. We denote the trivial bundle over M with fiber F by F_{M}. In the present work, "smooth" means C^∞. Following Helgason [22], the set of smooth functions on M, smooth vector fields on M and Pfaffian forms on M are denoted by $\mathfrak{F}(\mathsf{M})$, $\mathfrak{D}^1(\mathsf{M})$ and $\mathfrak{U}_1(\mathsf{M})$, respectively. If $\pi_E : E \to \mathsf{M}$ is a smooth vector fiber bundle over M then $\mathbb{O}_E$ will denote the zero section of E, that is, $\mathbb{O}_E = \{\mathbb{O}_p : p \in \mathsf{M}\}$, with $\mathbb{O}_p$ the zero vector of $E_p = \pi_E^{-1}[p]$, $p \in \mathsf{M}$. The set of smooth sections of $\pi_E : E \to \mathsf{M}$ is denoted by $\Gamma^\infty(E)$.

In the sequel, we recall some notions regarding the geometry of the tangent bundle $\mathsf{T}E$ of a smooth vector bundle E over M (see, for example [2,32] or [30]), which we will use later on.

Let $E \oplus_{\mathsf{M}} E$ denote the Whitney sum of $\pi_E : E \to \mathsf{M}$ with itself. The *vertical lift* is the map $\lambda^E : E \oplus_{\mathsf{M}} E \to \mathsf{T}E$ such that, for any $q \in \mathsf{M}$, $v_q \in E_q$, $\lambda^E_{v_q} = \lambda^E(v_q, \cdot) : E_q \to \mathsf{T}_{v_q}E$ is the tangent map at v_q of the inclusion $E_q \to E$, using the canonical identification $\mathsf{T}_{v_q}(E_q) \equiv E_q$. That is, for all $w_q \in E_q$, we have: $\lambda^E_{v_q}(w_q) = (T/\mathrm{d}t)|_{t=0}(v_q + tw_q)$.

The map λ^E is a smooth VB-monomorphism defined on the smooth vector bundle $\mathrm{pr}_1 : E \oplus_{\mathsf{M}} E \to E$ whose image is the *vertical sub-bundle* $\mathrm{Ver}(E) = \ker(\mathsf{T}\pi_E)$.

Let $\nabla : \Gamma^\infty(E) \to \Gamma^\infty(\mathsf{T}^*\mathsf{M} \otimes E)$ (or ∇^E, if there is a risk of confusion) denote a *connection* on $\pi_E : E \to \mathsf{M}$. That is, ∇ is an $\mathbb{R}$-linear map which satisfies the condition that, for any $f \in \mathfrak{F}(M)$ and any $\sigma \in \Gamma^\infty(E)$: $\nabla(f\sigma) = \mathrm{d}f \otimes \sigma + f\nabla\sigma$. The connection ∇ gives rise to a smooth VB-morphism $\mathrm{H}^E : E \oplus_{\mathsf{M}} \mathsf{TM} \to \mathsf{T}E$: for any $q \in \mathsf{M}$, $w_q \in E_q$ and $v_q \in \mathsf{T}_q\mathsf{M}$, choose any smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$, $t \mapsto \gamma(t)$, such that $(T\gamma/\mathrm{d}t)|_{t=0} = v_q$. Let $\tau_\gamma(t) : E_q \to E_{\gamma(t)}$ be the parallel transport along γ defined by the connection. Then the tangent vector at 0 of the smooth curve $t \in (-\varepsilon, \varepsilon) \to \tau_\gamma(t)w_q$ is independent on the choice of γ—it depends only on the pair (v_q, w_q). We denote it by $\mathrm{H}^E_{v_q}(w_q) = \mathrm{H}^E(v_q, w_q)$. H^E defines a VB-monomorphism of the smooth vector bundle $\mathrm{pr}_1 : E \oplus_{\mathsf{M}} \mathsf{TM} \to E$ into $\tau_E : \mathsf{T}E \to E$. Its image $\mathrm{Hor}(E)$ is the *horizontal sub-bundle* induced by the connection. $\mathrm{H}^E(v_q, w_q)$ is called the *horizontal lift* of w_q at v_q, and is the unique vector at $\mathrm{Hor}_{v_q}(E)$ which projects (through $\mathsf{T}\pi_E$) to the vector $w_q \in \mathsf{T}_q\mathsf{M}$.

The smooth vector bundle $\tau_E : \mathsf{T}E \to E$ is the Whitney sum $\mathrm{Hor}(E) \oplus_E \mathrm{Ver}(E)$ of its horizontal and vertical sub-bundles.

With a connection we can define the *connector* $\kappa_E : \mathsf{T}E \to E$, which is a VB-epimorphism from $\tau_E : \mathsf{T}E \to E$ to $\pi_E : E \to M$ such that for each $X_{v_q} \in \mathsf{T}E$, $\kappa_E(X_{v_q}) \in E_{v_q}$ is the unique vector which satisfies:

$$X_{v_q} = \mathrm{H}^E_{v_q}(\mathsf{T}\pi_E \cdot X_{v_q}) + \lambda^E_{v_q}(\kappa_E \cdot X_{v_q}). \tag{4}$$

Note that the restriction of the connector to the vertical bundle does not depend on the connection, since it is the inverse of the vertical lift $\kappa^V_E : \mathrm{Ver}(E) \to E$, $X_{v_q} \in \mathrm{Ver}_{v_q}E \mapsto (\lambda^E_{v_q})^{-1} \cdot X_{v_q}$.

The main significance of the preceding operators is that they allow us to work with objects in M and E instead of $\mathsf{T}E$. For example, let $u : (-\varepsilon, \varepsilon) \to E$ be a differentiable curve and $\gamma : (-\varepsilon, \varepsilon) \to \mathsf{M}$ be its projection on M, $\gamma = \pi_E \circ u$. Denoting by $\dot{u} := (Tu/\mathrm{d}t)$ the tangent vector field along u, we have $\kappa_E \cdot \dot{u} = \nabla_t u$, where ∇_t is the covariant derivative along γ associated to the connection ∇. Therefore, we have the following modified version of Eq. (4), which will be extensively used:

$$\dot{u} = \mathrm{H}_u(\dot{\gamma}) + \lambda_u(\nabla_t u).$$

For the sake of simplicity, from now on we will omit the "E" from the notation, using H, λ, κ instead of H^E, λ^E and κ_E, respectively, whenever there is no risk of confusion.

2.1. The fiber and parallel derivatives

Let $\pi_E : E \to \mathsf{M}$ and $\pi_F : F \to \mathsf{N}$ be smooth vector bundles over M and N, respectively, and let $b : E \to F$ be a smooth fiber bundle morphism over $\tilde{b} : \mathsf{M} \to \mathsf{N}$. That is, b, $\tilde{b}$ are smooth maps such that the following diagram is commutative:

$$\begin{tikzcd} E \arrow[r, "b"] \arrow[d, "\pi_E"'] \arrow[dr, phantom, "\circlearrowright"] & F \arrow[d, "\pi_F"] \\ \mathsf{M} \arrow[r, "\tilde{b}"'] & \mathsf{N} \end{tikzcd}$$

The concept of *fiber derivative* of b is well known (see, for example [1]); it is the fiber bundle morphism $\mathbb{F}b$ defined by

$$\mathbb{F}b : E \to \mathsf{L}(E, \tilde{b}^* F), \qquad v_q \mapsto \mathbb{F}b(v_q),$$

where $\tilde{b}^* F$ is the pull back vector bundle of F by $\tilde{b}$ and, for all $w_q \in E_q$:

$$\mathbb{F}b(v_q) \cdot w_q := \kappa^V_F \cdot \mathsf{T}b \cdot \lambda_{v_q}(w_q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} b(v_q + tw_q) \in F_{\tilde{b}(q)},$$

where $\mathrm{d}/\mathrm{d}t$ denotes the derivative of the curve $t \mapsto b(v_q + twq)$ on the linear space $F_{\tilde{b}(q)}$.

Given connections ∇^E and ∇^F on the vector bundles $\pi_E : E \to \mathsf{M}$ and $\pi_F : F \to \mathsf{N}$, respectively, we introduce in the following definition a dual concept to the fiber derivative of b.

Definition 1. The smooth fiber bundle morphism $\mathbb{P}b : E \to \mathsf{L}(\mathsf{TM}, \tilde{b}^* F)$ given by, for all $v_q \in E$ and all $z_q \in \mathsf{T}_q\mathsf{M}$:

$$\mathbb{P}b(v_q) \cdot z_q := \kappa_F \cdot \mathsf{T}b \cdot \mathrm{H}_{v_q}(z_q) \in F_{\tilde{b}(q)}$$

is called the *parallel derivative* of b.

The idea of introducing these objects is to use the globally defined "partial derivatives" $\mathbb{F}b$ and $\mathbb{P}b$ to compute the tangent map of b. The following formulae will be extensively used:

$$\mathsf{T}\pi_F \cdot \mathsf{T}b \cdot X_{v_q} = \mathsf{T}\tilde{b} \cdot \mathsf{T}\pi_E \cdot X_{v_q},$$
$$\kappa_F \cdot \mathsf{T}b \cdot X_{v_q} = \mathbb{F}b(v_q) \cdot \kappa_E \cdot X_{v_q} + \mathbb{P}b(v_q) \cdot \mathsf{T}\pi_F \cdot X_{v_q},$$

so that, given a curve γ in M and a differentiable section X of E along γ, we have

$$\nabla_t^F (b \circ X) = \mathbb{F}b(X) \cdot \nabla_t^E X + \mathbb{P}b(X) \cdot \dot{\gamma}.$$

Besides, the connections on the vector bundles E and F canonically induce a connection on the smooth vector bundle $\mathsf{L}(E, \tilde{b}^* F)$ over M. If a connection on the tangent bundle TM is given, we also have a canonically induced connection on $\mathsf{L}(\mathsf{TM}, \tilde{b}^* F)$. Hence, we can take the fiber and parallel derivatives of the smooth fiber bundle morphisms $\mathbb{F}b : E \to \mathsf{L}(E, \tilde{b}^* F)$ and $\mathbb{P}b : E \to \mathsf{L}(\mathsf{TM}, \tilde{b}^* F)$, yielding smooth fiber bundle morphisms:

$$\mathbb{F}\mathbb{F}b : E \to \mathsf{L}(E, \mathsf{L}(E, \tilde{b}^* F)) \equiv \mathsf{L}(E \otimes E, \tilde{b}^* F),$$
$$\mathbb{P}\mathbb{F}b : E \to \mathsf{L}(\mathsf{TM}, \mathsf{L}(E, \tilde{b}^* F)) \equiv \mathsf{L}(\mathsf{TM} \otimes E, \tilde{b}^* F),$$
$$\mathbb{F}\mathbb{P}b : E \to \mathsf{L}(E, \mathsf{L}(\mathsf{TM}, \tilde{b}^* F)) \equiv \mathsf{L}(E \otimes \mathsf{TM}, \tilde{b}^* F),$$
$$\mathbb{P}\mathbb{P}b : E \to \mathsf{L}(\mathsf{TM}, \mathsf{L}(\mathsf{TM}, \tilde{b}^* F)) \equiv \mathsf{L}(\mathsf{TM} \otimes \mathsf{TM}, \tilde{b}^* F).$$

Proposition 1. *Given $v_q \in E$, we have the following relations*:

1. $\mathbb{F}^2 b(v_q) \cdot (w_q, z_q) = \mathbb{F}^2 b(v_q) \cdot (z_q, w_q)$ *for all* $w_q, z_q \in E_q$;
2. $\mathbb{F}\mathbb{P}b(v_q) \cdot (w_q, z_q) = \mathbb{P}\mathbb{F}b(v_q) \cdot (z_q, w_q)$ *for all* $w_q \in E_q$, $z_q \in \mathsf{T}_q\mathsf{M}$;
3. $\mathbb{P}^2 b(v_q) \cdot (w_q, z_q) = \mathbb{P}^2 b(v_q) \cdot (z_q, w_q) + \mathbb{F}b(v_q) \cdot \mathsf{R}^E(z_q, w_q) \cdot v_q + \mathsf{R}^F(\mathsf{T}\tilde{b} \cdot w_q, \mathsf{T}\tilde{b} \cdot z_q) \cdot b(v_q)$ *for all* $w_q, z_q \in \mathsf{T}_q\mathsf{M}$, *where* R^E *and* R^F *are the curvature tensors of* ∇^E *and* ∇^F, *respectively*.

Finally, given $f \in \mathfrak{F}(E)$, we consider the smooth fiber bundle morphism $\tilde{f} : E \to \mathbb{R}_\mathsf{M}$, defined by $v_q \mapsto (q, f(q))$. Let us endow the vector bundle $\mathbb{R}_\mathsf{M}$ with the trivial connection, that is, defined by $\nabla e_1 = 0$, where $e_1 : x \in \mathsf{M} \mapsto 1 \in \mathbb{R}_x$. Then, for all $v_q \in E$ and $X_{v_q} \in \mathsf{T}_{v_q}E$, we have

$$\mathrm{d}f(v_q) \cdot X_{v_q} = \kappa_{\mathbb{R}_\mathsf{M}} \cdot \mathsf{T}_{v_q}\tilde{f} \cdot X_{v_q} = \mathbb{F}\tilde{f}(v_q) \cdot \kappa_E \cdot X_{v_q} + \mathbb{P}\tilde{f}(v_q) \cdot \mathsf{T}\pi_E \cdot X_{v_q}.$$

We will omit henceforth the "$\sim$" from the notation, tacitly identifying f with $\tilde{f}$, and we will employ this formula to compute $\mathrm{d}f$.

2.2. The geometry of the constraint manifold

In this subsection, we give examples and describe some notation and some facts concerning the geometry of the constraint manifold.

Definition 2 (Marle). A *constraint* on M is a smooth embedded submanifold $\mathcal{C}$ of TM such that the restriction to $\mathcal{C}$ of the projection of the tangent bundle $\tau_{\mathsf{M}} : \mathsf{TM} \to \mathsf{M}$, henceforth denoted by $\pi_{\mathcal{C}}$, is a submersion. The constraint is said to be *linear* if $\mathcal{C}$ is a smooth vector sub-bundle of TM; we use the symbol $\mathcal{D}$ to denote linear constraints.

The hypothesis of $\pi_{\mathcal{C}} : \mathcal{C} \to \mathsf{M}$ being a submersion ensures that, for all *admissible velocity* $v_q \in \mathcal{C}$, there exists a motion compatible with the constraint $\gamma : (-\varepsilon, \varepsilon) \to \mathsf{M}$ whose initial velocity $\dot{\gamma}(0)$ coincides with v_q; this is a necessary condition for the existence of second-order vector fields tangent to $\mathcal{C}$. To check its validity, given $v_q \in \mathcal{C}$, the fact of $\pi_{\mathcal{C}} : \mathcal{C} \to \mathsf{M}$ being a submersion implies the existence of a local smooth section X of $\pi_{\mathcal{C}}$, defined on an open set $\mathcal{U} \subset \mathsf{M}$ containing q and such that $X(q) = v_q$; an integral curve of the vector field X with initial condition q is a motion compatible with $\mathcal{C}$ with initial velocity v_q.

Given $q \in \mathsf{M}$, we denote by $\mathcal{C}_q$ the embedded submanifold $\pi_{\mathcal{C}}^{-1}[q] \subset \mathsf{T}_q\mathsf{M}$. This is indeed a submanifold of $\mathsf{T}_q\mathsf{M}$, since it is a submanifold of TM (because it is a submanifold of $\mathcal{C}$, by the hypothesis of $\pi_{\mathcal{C}} : \mathcal{C} \to \mathsf{M}$ being a submersion, and $\mathcal{C}$ is a submanifold of TM) and it is contained in the embedded submanifold $\mathsf{T}_q\mathsf{M}$ of TM.

The following proposition is used in the construction of some examples.

Proposition 2. *Let* S *be a smooth vector bundle over* M, $f : \mathsf{TM} \to \mathsf{S}$ *a smooth fiber bundle morphism and* $\mathcal{C} := f^{-1}[\mathbb{O}_{\mathsf{S}}]$. *The following conditions are equivalent*:

(i) *f is transversal to the null section* $\mathbb{O}_{\mathsf{S}}$ *and* $\tau_{\mathsf{M}}|_{\mathcal{C}} : \mathcal{C} \to \mathsf{M}$ *is a submersion* (*so that* $\mathcal{C}$ *is a constraint, closed in* TM);
(ii) $(\forall v_q \in \mathsf{TM})\mathbb{F}f(v_q) : \mathsf{T}_q\mathsf{M} \to \mathsf{S}_q$ *is surjective*.

Example 1.

(a) The simplest example of a constraint that is not linear is provided by an affine constraint. In this case $\mathcal{C}$ is an affine sub-bundle of TM: given a pair $(\mathcal{D}, X_a)$, where $\mathcal{D}$ is a smooth vector sub-bundle and $X_a \in \mathfrak{D}^1(\mathsf{M})$, we take, for all $q \in \mathsf{M}$, $\mathcal{C}_q = \mathcal{D}_q + X_a(q)$.
(b) (Carathéodory). Let $\mathsf{M} = \mathbb{R}^2$ and denote by $x = (x_1, x_2) \in \mathbb{R}^2$ the Cartesian coordinates of the point $x \in \mathbb{R}^2$ and by $v = (v_1, v_2) \in \mathbb{R}^2$ the corresponding velocity vector. Then, we define $f : \mathsf{TM} \to \mathbb{R}_{\mathsf{M}}$ by $f_x(v_1, v_2) = v_2 - \sqrt{1 + v_1^2}$ and apply Proposition 2.
(c) (Marle's [36] servomechanism). This example can be viewed as the model of a control system formed by a rod on a vertical plane and an actuator which communicates motion to its lower extremity along a fixed horizontal line, in order to manipulate the rod in a certain way—see Fig. 1.

We put $\mathsf{M} = \mathbb{R} \times S^1$ and $f : \mathsf{TM} \equiv \mathbb{R}^2_{\mathsf{M}} \to \mathbb{R}_{\mathsf{M}}$ given by $f(x, \theta, \dot{x}, \dot{\theta}) = \dot{x} - h(x, \theta, \dot{\theta})$, where $h : \mathbb{R}_{\mathsf{M}} \to \mathbb{R}$ is a smooth function. Then, applying Proposition 2, $\mathcal{C} := f^{-1}[\mathbb{O}_{\mathbb{R}_{\mathsf{M}}}]$ is a constraint.

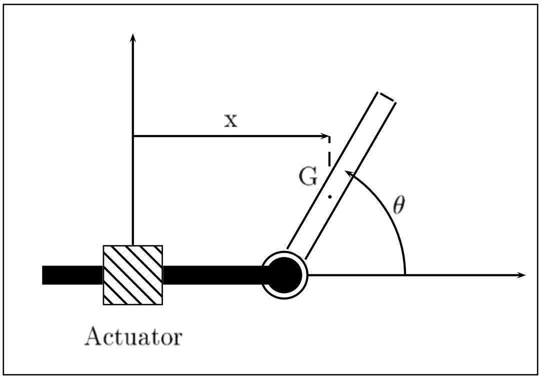

Fig. 1. A servomechanism.

(d) (Isokinetic dynamics). Let $e > 0$. We define the constraint applying Proposition 2, with $f : \mathsf{TM} \to \mathbb{R}_{\mathsf{M}}$ given by $f(v_q) = (1/2)\langle v_q, v_q\rangle - e$, see [19,23,45,56].

(e) (Benenti's example). In [17] a non-linear, quadratic homogeneous constraint in the velocities is proposed. It requires that two points in the plane have parallel velocities to each other. We put $M = \mathbb{R}^4$, and denoting by $x = (x_1, x_2, x_3, x_4)$ the combined Cartesian coordinates of the two points and by $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ the corresponding vector of the velocities, we define

$$f_x(v_1, v_2, v_3, v_4) = \det\begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} = v_1v_4 - v_2v_3.$$

Then $\mathcal{C} := f^{-1}[\mathbb{O}_{\mathbb{R}_{\mathsf{M}}}] \setminus \mathbb{O}_{\mathsf{TM}}$ is a *cone*. That is to say, given $v_q \in \mathcal{C}$, then $(\forall t > 0) tv_q \in \mathcal{C}$. Note that it is necessary to remove the null section $\mathbb{O}_{\mathsf{TM}}$ from $f^{-1}[\mathbb{O}_{\mathbb{R}_{\mathsf{M}}}]$ (in other words, we impose the additional condition that the velocities of the points cannot be simultaneously null), in order for $\mathcal{C}$ to be a smooth submanifold of TM.

Since $\pi_{\mathcal{C}} : \mathcal{C} \to \mathsf{M}$ is a submersion, $\mathsf{T}\pi_{\mathcal{C}} : \mathsf{T}\mathcal{C} \to \mathsf{TM}$ is a smooth vector bundle epimorphism; then $\ker \mathsf{T}\pi_{\mathcal{C}}$ is a smooth vector sub-bundle of $\mathsf{T}\mathcal{C}$, denoted henceforth by $\mathrm{Ver}(\mathcal{C})$, and called the *vertical sub-bundle* of $\mathsf{T}\mathcal{C}$. This sub-bundle is integrable; indeed, for all $v_q \in \mathcal{C}$, we have $\mathsf{T}_{v_q}(\mathcal{C}_q) = \mathrm{Ver}_{v_q}(\mathcal{C})$. Given $v_q \in \mathcal{C}$, we call $C_{v_q} := \kappa^V \cdot \mathrm{Ver}_{v_q}(\mathcal{C}) \subset \mathsf{T}_q\mathsf{M}$ the *subspace of virtual velocities* (following the nomenclature of [5]) at v_q; C_{v_q} is the subspace of $\mathsf{T}_q\mathsf{M}$ which is the image of the tangent map at v_q of the inclusion $\mathcal{C}_q \to \mathsf{T}_q\mathsf{M}$.

Denoting by $\iota_{\mathcal{C}} : \mathcal{C} \to \mathsf{TM}$ the inclusion, $\mathsf{T}\mathcal{C}$ is a vector sub-bundle of the pull back vector bundle $\iota_{\mathcal{C}}^*\mathsf{TTM}$, also denoted by $\mathsf{TTM}|_{\mathcal{C}}$. Let us endow the vertical bundle $\mathrm{Ver}(\mathsf{TM})$ with the metric tensor induced by the metric $\mathfrak{g}$ of M through the vertical lift, i.e. such that $(\forall v_q \in \mathsf{TM})\lambda_{v_q} : \mathsf{T}_q\mathsf{M} \to \mathrm{Ver}_{v_q}(\mathsf{TM})$ is a linear isometry. Since $\mathrm{Ver}(\mathcal{C})$ is a vector sub-bundle of the *pull back* $\iota_{\mathcal{C}}^*\,\mathrm{Ver}(\mathsf{TM})$, it makes sense to consider the orthogonal sub-bundle W of $\mathrm{Ver}(\mathcal{C})$ in $i_{\mathcal{C}}^*\,\mathrm{Ver}(\mathsf{TM})$. That is to say, for all $v_q \in \mathcal{C}$, $W_{v_q} := \mathrm{Ver}_{v_q}(\mathcal{C})^{\perp}$ is the orthogonal complement of $\mathrm{Ver}_{v_q}(\mathcal{C})$ in $\mathrm{Ver}_{v_q}(\mathsf{TM})$. The vector bundle $\pi_W : W \to \mathcal{C}$ is called the *projection bundle* (W is the pull back by the Legendre transformation μ of Marle's [35] projection bundle W

over $D := \mu(\mathcal{C}) \subset \mathsf{T}^*\mathsf{M}$) on $\mathcal{C}$, induced by $\mathfrak{g}$. For all $v_q \in \mathcal{C}$, the restrictions of the vertical lift at v_q to C_{v_q} and to its orthogonal complement are linear isometries: $\lambda_{v_q} : C_{v_q} \to \mathrm{Ver}_{v_q}(\mathcal{C})$ and $\lambda_{v_q} : C_{v_q}^{\perp} \to W_{v_q}$. We denote the orthogonal projections $\mathsf{T}_q\mathsf{M} \to C_{v_q}$ and $\mathsf{T}_q\mathsf{M} \to C_{v_q}^{\perp}$ by $\mathcal{P}(v_q)$ and $\mathcal{P}^{\perp}(v_q)$, respectively.

By the construction of W, we have the Whitney sum decomposition $i_{\mathcal{C}}^* \mathrm{Ver}(\mathsf{TM}) = \mathrm{Ver}(\mathcal{C}) \oplus_{\mathcal{C}} W$. Besides, we also have the Whitney sum given by the following proposition [35].

Proposition 3. *In the above situation, the following Whitney sum decomposition holds*:

$$i_{\mathcal{C}}^*(\mathsf{TTM}) = \mathsf{T}\mathcal{C} \underset{\mathcal{C}}{\oplus} W. \tag{5}$$

We denote by $P_{\mathcal{C}}$ and P_W the projections on the first and second factor of (5), respectively. Note that we have made use of the hypothesis of $\pi_{\mathcal{C}}$ being a submersion to construct the above splittings of $\mathsf{TTM}|_{\mathcal{C}}$ and $\mathrm{Ver}(\mathsf{TM})|_{\mathcal{C}}$.

Let us now consider the Levi–Civita connection ∇ of $(\mathsf{M}, \mathfrak{g})$, and the corresponding horizontal sub-bundle $\mathrm{Hor}(\mathsf{TM}) \subset \mathsf{TTM}$. We denote by $\mathrm{Hor}(\mathcal{C})$ the image by $P_{\mathcal{C}}$ of $\iota_{\mathcal{C}}^*\mathrm{Hor}(\mathsf{TM})$. We call $\mathrm{Hor}(\mathcal{C})$ the *horizontal sub-bundle* of $\mathsf{T}\mathcal{C}$, induced by $\mathfrak{g}$, and we have the following Whitney sum decomposition:

$$\mathsf{T}\mathcal{C} = \mathrm{Hor}(\mathcal{C}) \underset{\mathcal{C}}{\oplus} \mathrm{Ver}(\mathcal{C}). \tag{6}$$

We denote by $P_H^{\mathcal{C}} : \mathsf{T}\mathcal{C} \to \mathrm{Hor}(\mathcal{C})$ and $P_V^{\mathcal{C}} : \mathsf{T}\mathcal{C} \to \mathrm{Ver}(\mathcal{C})$ the projections on the first and second factor of (6), respectively. Given $v_q \in \mathcal{C}$, we define the vertical and horizontal lifts in $\mathsf{T}\mathcal{C}$, $\lambda_{v_q}^{\mathcal{C}} := \lambda_{v_q} \circ \mathcal{P}_{v_q} = P_{\mathcal{C}} \circ \lambda_{v_q} : \mathsf{T}_q\mathsf{M} \to \mathrm{Ver}_{v_q}(\mathcal{C})$ and $\mathrm{H}_{v_q}^{\mathcal{C}} := (\mathsf{T}\tau_{\mathsf{M}}|_{\mathrm{Hor}_{v_q}(\mathcal{C})})^{-1} = P_{\mathcal{C}} \circ \mathrm{H}_{v_q} : \mathsf{T}_q\mathsf{M} \to \mathrm{Hor}_{v_q}(\mathcal{C})$.

Note that, for all $v_q \in \mathcal{C}$, $\mathrm{H}_{v_q}^{\mathcal{C}} : \mathsf{T}_q\mathsf{M} \to \mathrm{Hor}_{v_q}(\mathcal{C})$ and $\lambda_{v_q}^{\mathcal{C}}|_{C_{v_q}} : C_{v_q} \to \mathrm{Ver}_{v_q}(\mathcal{C})$ are linear isomorphisms.

In the case of a linear constraint $\mathcal{D}$, the Whitney sum decomposition (6) coincide with the one induced by the connection on $\mathcal{D}$ defined by $\nabla^{\mathcal{D}} : \mathfrak{D}^1(\mathsf{M}) \times \Gamma^{\infty}(\mathcal{D}) \to \Gamma^{\infty}(\mathcal{D})$, $\nabla_X^{\mathcal{D}} Y := \mathcal{P}_{\mathcal{D}} \cdot \nabla_X Y$, where $\mathcal{P}_{\mathcal{D}} : \mathsf{TM} \to \mathcal{D}$ is the orthogonal projection. In that case, given $v_q \in \mathcal{D}$, $\lambda_{v_q}^{\mathcal{D}}$ and $\mathrm{H}_{v_q}^{\mathcal{D}}$ are the usual vertical and horizontal lifts at v_q, and we have $C_{v_q} = \mathcal{D}_q$, $W_{v_q} = \lambda_{v_q}(\mathcal{D}_q^{\perp})$, so that $P_{\mathcal{D}} = \mathsf{T}\mathcal{P}_{\mathcal{D}} : \mathsf{TTM}|_{\mathcal{C}} \to \mathcal{D}$.

We define next the fiber and parallel derivatives for maps $\mathcal{C} \to E$, where $\pi_E : E \to \mathsf{M}$ a smooth vector bundle, which preserve fibers. That is the case, for example, of the maps $\mathcal{P}, \mathcal{P}^{\perp} : \mathcal{C} \to \mathsf{L}(\mathsf{TM}, \mathsf{TM})$.

Definition 3. Let $\pi_E : E \to \mathsf{M}$ be a smooth vector bundle, endowed with a connection ∇^E, and $f : \mathcal{C} \to E$ a smooth map such that, for all $q \in \mathsf{M}$, $f(\mathcal{C}_q) \subset E_q$. We define the *fiber derivative* $\mathbb{F}f : \mathcal{C} \to \mathsf{L}(\mathsf{TM}, E)$ and the *parallel derivative* $\mathbb{P}f : \mathcal{C} \to \mathsf{L}(\mathsf{TM}, E)$ by, for all $v_q \in \mathcal{C}$:

$$\mathbb{F}f(v_q) := \kappa_E \circ \mathsf{T}_{v_q} f \circ \lambda_{v_q}^{\mathcal{C}} \in \mathsf{L}(\mathsf{T}_q\mathsf{M}, E_q),$$

$$\mathbb{P}f(v_q) := \kappa_E \circ \mathsf{T}_{v_q} f \circ \mathrm{H}_{v_q}^{\mathcal{C}} \in \mathsf{L}(\mathsf{T}_q\mathsf{M}, E_q).$$

Therefore, given $v_q \in \mathcal{C}$ and $X_{v_q} \in \mathsf{T}_{v_q}\mathcal{C}$, we have

$$\kappa_E \cdot \mathsf{T}_{v_q} f \cdot X_{v_q} = \mathbb{F} f(v_q) \cdot \kappa \cdot X_{v_q} + \mathbb{P} f(v_q) \cdot \mathsf{T}\pi_{\mathcal{C}} \cdot X_{v_q}$$

and $\mathbb{F} f(v_q) \cdot \kappa \cdot X_{v_q} = \mathbb{F} f(v_q) \cdot \mathcal{P}_{v_q} \cdot \kappa \cdot X_{v_q}$, i.e. $C^{\perp}_{v_q} \subset \ker \mathbb{F} f(v_q)$.

By a previous observation, in the linear case these derivatives coincide with the fiber and parallel derivatives defined in the previous subsection, endowing $\mathcal{D}$ with the connection $\nabla^{\mathcal{D}}$.

As a final remark, given $f \in \mathfrak{F}(\mathsf{TM})$, we use the notation $\mathbb{F}^\sharp f$ and $\mathbb{P}^\sharp f$ to denote, respectively, the maps $\mathfrak{g}^\sharp \circ \mathbb{F} f : \mathsf{TM} \to \mathsf{TM}$ and $\mathfrak{g}^\sharp \circ \mathbb{P} f : \mathsf{TM} \to \mathsf{TM}$, where $\mathfrak{g}^\sharp$ is the inverse of the Legendre transformation $\mathfrak{g}^\flat : \mathsf{TM} \to \mathsf{T}^*\mathsf{M}$ induced by the metric tensor $\mathfrak{g}$.

2.2.1. Second-order vector fields on $\mathcal{C}$

Definition 4. Given a constraint $\mathcal{C} \subset \mathsf{TM}$, the subset $\mathfrak{P}(\mathcal{C}) := \mathsf{T}\mathcal{C} \cap J^2(\mathsf{M})$ of $\mathsf{T}\mathcal{C}$ is called *holonomic prolongation* of $\mathcal{C}$ (see [40]). Here, $J^2(\mathsf{M}) := \{z \in \mathsf{T}(\mathsf{TM}) | \tau_{\mathsf{TM}} z = \mathsf{T}\tau_{\mathsf{M}}(z)\}$ is the 2-jets affine sub-bundle of TM.

The following proposition shows that $\mathfrak{P}(\mathcal{C})$ is an affine sub-bundle of $\mathsf{T}\mathcal{C}$.

Proposition 4. *With the same notation, $\tau_{\mathsf{TM}}|_{\mathfrak{P}(\mathcal{C})} : \mathfrak{P}(\mathcal{C}) \to \mathcal{C}$ is a smooth affine sub-bundle of $\mathsf{T}\mathcal{C}$. More precisely, for each $v_q \in \mathcal{C}$, $\mathfrak{P}_{v_q}(\mathcal{C})$ is the affine subspace $P_{\mathcal{C}} \cdot \mathsf{S}(v_q) + \mathrm{Ver}_{v_q}(\mathcal{C})$ of $\mathsf{T}_{v_q}\mathcal{C}$, where S is the geodesic spray of $(\mathsf{M}, \mathfrak{g})$.*

Proof. We have $\mathfrak{P}(\mathcal{C}) = \mathsf{T}\mathcal{C} \cap J^2(\mathsf{M}) = \{X_{v_q} \in \mathsf{T}\mathcal{C} | \mathsf{T}\tau_{\mathsf{M}} \cdot X_{v_q} = v_q\}$. Hence, given $v_q \in \mathcal{C}$ and $X_{v_q} \in \mathfrak{P}_{v_q}(\mathcal{C})$, it follows that $\mathsf{T}\tau_{\mathsf{M}} \cdot X_{v_q} = v_q = \mathsf{T}\tau_{\mathsf{M}} \cdot P_{\mathcal{C}} \cdot \mathsf{S}(v_q)$, therefore $X_{v_q} - P_{\mathcal{C}} \cdot \mathsf{S}(v_q) \in \mathsf{T}_{v_q}\mathcal{C} \cap \mathrm{Ver}_{v_q}(\mathsf{TM}) = \mathrm{Ver}_{v_q}(\mathcal{C})$, that is to say, $X_{v_q} \in P_{\mathcal{C}} \cdot \mathsf{S}(v_q) + \mathrm{Ver}_{v_q}(\mathcal{C})$. On the other hand, given $X_{v_q} \in P_{\mathcal{C}} \cdot \mathsf{S}(v_q) + \mathrm{Ver}_{v_q}(\mathcal{C})$, we have $X_{v_q} \in \mathsf{T}_{v_q}\mathcal{C}$ e $\mathsf{T}\tau_{\mathsf{M}} \cdot X_{v_q} = \mathsf{T}\tau_{\mathsf{M}} \cdot P_{\mathcal{C}} \cdot \mathsf{S}(v_q) = v_q$, thus $X_{v_q} \in \mathfrak{P}_{v_q}(\mathcal{C})$. We have then shown $\mathfrak{P}_{v_q}(\mathcal{C}) = P_{\mathcal{C}} \cdot \mathsf{S}(v_q) + \mathrm{Ver}_{v_q}(\mathcal{C})$, for all $v_q \in \mathcal{C}$. □

Definition 5. We say that $X \in \mathfrak{D}^1(\mathcal{C})$ is a *second-order vector field on $\mathcal{C}$* if it is a section of the holonomic prolongation $\mathfrak{P}(\mathcal{C})$.

Note that, given a second-order vector field on $\mathcal{C}$, $X \in \Gamma^\infty(\mathfrak{P}(\mathcal{C}))$, since $(\forall v_q \in \mathcal{C})\mathsf{T}\tau_{\mathsf{M}} \cdot X(v_q) = v_q$ the integral curves of X are of the form $T\gamma/\mathrm{d}t$, where γ is a smooth horizontal curve on M.

3. Statement of the main results

In this section we state and describe the main results of the paper. We use the notation from Section 2 and, unless otherwise stated, we consider a fixed constrained mechanical system $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$.

3.1. Gauss' principle and the d'Alembert–Chetaev trajectories

In this subsection we define the d'Alembert–Chetaev trajectories of $(\mathsf{M}, \mathrm{K}, \mathcal{F})$ through Gauss' principle of least constraint. Firstly, we introduce the concept of *admissible reaction*.

Definition 6. We say that a continuous map $R : \mathcal{C} \to \mathsf{TM}$ is an *admissible reaction field* for the constrained mechanical system $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$ if it is fiber preserving and if there exists a second-order vector field $X_{\mathcal{C}}^{R}$ on $\mathcal{C}$ whose maximal integral curves with fixed initial condition exist and are unique, and whose base integral curves are solutions of Newton's equation with reaction term (3).

We denote by $\mathfrak{R}$ the set of all admissible reaction fields for $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$. If $R \in \mathfrak{R}$, we call the base integral curves of $X_{\mathcal{C}}^{R}$ the *trajectories* of the constrained mechanical system $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$, induced by the admissible reaction R.

Remark 1. Note that, if R is an admissible reaction field for $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$, then $X_{\mathcal{C}}^{R}$ is univocally determined by R. In fact, taking vertical lifts in (3), it follows that $X_{\mathcal{C}}^{R}$ must be given by

$$\mathsf{S}(v_q) + \lambda_{v_q}\{\mathcal{F}^{\sharp}(v_q) + R(v_q)\}, \tag{7}$$

where S is the geodesic spray of $(\mathsf{M}, \mathfrak{g})$.

Proposition 5. *The admissible reaction fields for* $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$ *are the continuous fiber preserving maps* $R : \mathcal{C} \to \mathsf{TM}$ *which satisfy, for all* $v_q \in \mathcal{C}$:

$$\mathcal{P}_{v_q}^{\perp} \cdot (R(v_q)) = -\kappa \cdot P_W \cdot \mathsf{S}(v_q) - \mathcal{P}_{v_q}^{\perp} \cdot (\mathcal{F}^{\sharp}(v_q)) \tag{8}$$

and such that the uniqueness and existence property holds for the integral curves of the vector field $X_{\mathcal{C}}^{R}$, *where* W *is the projection bundle on* $\mathcal{C}$.

Proof. It is enough to check that, given a continuous fiber preserving map $R : \mathcal{C} \to \mathsf{TM}$, $X_{\mathcal{C}}^{R}(\mathcal{C}) \subset \mathsf{T}\mathcal{C}$ if, and only if, Eq. (8) holds (where $X_{\mathcal{C}}^{R}$ is given by Eq. (7)). Indeed, given $v_q \in \mathcal{C}$, $X_{\mathcal{C}}^{R}(v_q) \in \mathsf{T}_{v_q}\mathcal{C}$ if, and only if $0 = \kappa \cdot P_W \cdot X_{\mathcal{C}}^{R}(v_q) = \kappa \cdot P_W \cdot \lambda_{v_q}(\mathcal{F}^{\sharp}(v_q) + R(v_q)) + \kappa \cdot P_W \cdot \mathrm{H}_{v_q} v_q$, what is equivalent to Eq. (8). □

At this point we should note that there are, in general, many possible choices for the admissible reaction force. In a "physical" constraint, however, one could reasonably expect that the reaction should be determined by the external force and that each motion of the system be uniquely determined by its initial velocity. The reason why this does not happen in our model is the fact that, what we call "constraint" is, in fact, a "kinematic constraint". A "physical" constraint is characterized not only by kinematic constraints, but also by a "dynamical" or "phenomenological" law [6]—something that in our model would correspond to a rule to determine the admissible reaction field. To illustrate this fact, we show two admissible reactions for the constrained mechanical system of the example of the servomechanism.

Example 2. In the example of the servomechanism (Example 1(c)), let us consider:

$$\mathrm{K}(x, \theta, \dot{x}, \dot{\theta}) = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}(I + ml^2 \cos^2\theta)\dot{\theta}^2, \qquad \mathrm{V}(x, \theta) = mgl \sin\theta,$$

where m is the mass of the rod, I the moment of inertia of the rod with respect to its barycentre, l the distance from the articulation with the actuator on the Ox axis to the barycentre of the rod and g the acceleration of gravity. For the sake of simplicity, let us put $m = I = l = g = 1$.

A direct computation shows that the following maps are admissible reactions for $(\mathsf{M}, \mathrm{K}, \mathrm{V}, \mathcal{C})$:

$$R^A(x, \theta, \dot{x}, \dot{\theta}) = \left(\frac{\mathbb{F}h(\dot{\theta}) \cos\theta}{(1 + \mathbb{F}h(\dot{\theta})^2)(1 + \cos^2\theta)} + \frac{\mathbb{P}h(\dot{\theta}) \cdot (\dot{x}, \dot{\theta})}{1 + \mathbb{F}h(\dot{\theta})^2}\right)(\partial_x - \mathbb{F}h(\dot{\theta})\partial_\theta)$$

and

$$R(x, \theta, \dot{x}, \dot{\theta}) = \left(\frac{\mathbb{F}h(\dot{\theta}) \cos\theta}{1 + \cos^2\theta} + \mathbb{P}h(\dot{\theta}) \cdot (\dot{x}, \dot{\theta})\right)\partial_x.$$

The reaction R^A has minimal intensity among all admissible reaction fields; the solutions of Newton's equation (3) with reaction term R^A are the d'Alembert–Chetaev trajectories of the constrained mechanical system $(\mathsf{M}, \mathrm{K}, \mathrm{V}, \mathcal{C})$ (see Theorem 1 and Definition 8). The reaction R, on the other hand, is the admissible reaction which corresponds to the physical hypothesis of the articulation being frictionless and of the actuator introducing no torque on the articulation (i.e. R has no term in ∂_θ).

Let $X_{\mathcal{F}} : \mathsf{TM} \to \mathsf{T}(\mathsf{TM})$ be the GMA vector field of the unconstrained mechanical system $(\mathsf{M}, \mathrm{K}, \mathcal{F})$ (i.e. the second-order vector field whose base integral curves are the solutions of Newton's equation (2)). That is to say, $X_{\mathcal{F}} : v_q \in \mathsf{TM} \mapsto \mathsf{S}(v_q) + \lambda_{v_q}\{\mathcal{F}^\sharp(v_q)\}$. Using the Whitney sum decomposition $\mathsf{TTM}|_{\mathcal{C}} = \mathsf{T}\mathcal{C} \oplus_{\mathcal{C}} W$, the restriction of $X_{\mathcal{F}}$ to the constraint manifold $\mathcal{C}$ splits into a sum $X_{\mathcal{F}}|_{\mathcal{C}} = X_{\mathcal{C}} + X_W$, where $X_{\mathcal{C}}$ is a smooth second-order vector field on $\mathcal{C}$ and X_W a smooth section of the projection bundle W.

Definition 7. Using the notation above, we call the second-order vector field $X_{\mathcal{C}} \in \mathfrak{D}^1(\mathcal{C})$ the Gibbs–Maggi–Appell (GMA) vector field of the constrained mechanical system $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$.

We show in Theorem 1 that the GMA vector field of $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$ is induced by an admissible reaction R^A that has the remarkable property of minimizing the intensity of the admissible reactions.

3.1.1. Gauss' principle of least constraint

For a linear constraint $\mathcal{D}$, there exists a unique admissible reaction $R^A : \mathcal{D} \to \mathsf{TM}$ satisfying the so-called "d'Alembert's principle". That is to say, R^A is orthogonal to the constraint, in the sense that $(\forall v_q \in \mathcal{D}) R^A(v_q) \perp \mathcal{D}_q$, i.e. $(\forall w_q \in \mathcal{D}_q)\langle R^A(v_q), w_q\rangle = 0$.

On the other hand, it can be easily checked that, if $\mathcal{C}$ is a non-linear constraint, in general there is no admissible reaction satisfying the above condition, i.e. orthogonal to the constraint manifold on each fiber of TM. However, we prove in the next theorem that there exists a unique admissible reaction satisfying Gauss' principle of least constraint, in the sense that it has minimal intensity among all admissible reaction fields. If the constraint is linear, this admissible reaction coincides with that given by d'Alembert's principle; in this sense, we can consider Gauss' principle of least constraint as a generalization of d'Alembert's principle for constraints which are non-linear in the velocities.

Theorem 1 (Gauss' principle of least constraint). *There exists a unique smooth admissible reaction field $R^A \in \mathfrak{R}$ such that, for all $v_q \in \mathcal{C}$:*

$$\|R^A(v_q)\| = \min_{R\in\mathfrak{R}} \|R(v_q)\|. \tag{9}$$

Moreover, the solutions of Newton's equation (3) with reaction term R^A coincide with the base integral curves of the GMA vector field $X_{\mathcal{C}}$—in particular they are smooth—i.e. $X_{\mathcal{C}}$ coincides with the second-order vector field $X_{\mathcal{C}}^{R^A}$ induced by the admissible reaction R^A.

Definition 8. The base integral curves of the GMA vector field $X_{\mathcal{C}}$ (or, equivalently, by the previous theorem, the solutions of Newton's equation (3) with reaction term R^A) are called the *d'Alembert–Chetaev trajectories of* $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$.

3.1.2. Gibbs–Appell equations

To close this subsection, we show that the d'Alembert–Chetaev trajectories are solutions of the so-called *Gibbs–Appell equations* [43] of the mechanical system.

We define the *Sasaki metric tensor* $\mathfrak{g}_{\mathsf{TM}}$ on TM (see [46,47,57]) by, for all $v_q \in \mathsf{TM}$, $X_{v_q}, Y_{v_q} \in \mathsf{T}_q\mathsf{M}$: $\langle X_{v_q}, Y_{v_q}\rangle_{\mathsf{T}_{v_q}\mathsf{TM}} := \langle \kappa \cdot X_{v_q}, \kappa \cdot Y_{v_q}\rangle_{\mathsf{T}_q\mathsf{M}} + \langle \mathsf{T}\tau_{\mathsf{M}} \cdot X_{v_q}, \mathsf{T}\tau_{\mathsf{M}} \cdot Y_{v_q}\rangle_{\mathsf{T}_q\mathsf{M}}$, where $\kappa : \mathsf{TTM} \to \mathsf{TM}$ is the connector induced by the Levi–Civita connection of $(\mathsf{M}, \mathfrak{g})$. With this metric the horizontal and vertical spaces are orthogonal to each other. With the Sasaki metric $\mathfrak{g}_{\mathsf{TM}}$, we define the Gibbs–Appell function of $(\mathsf{M}, \mathrm{K}, \mathcal{F})$.

Definition 9. Given a mechanical system $(\mathsf{M}, \mathrm{K}, \mathcal{F})$, we define its *Gibbs–Appell function* $\mathfrak{G} : \mathsf{TTM} \to \mathbb{R}$ (see [34,43]) by, for all $X_{v_q} \in \mathsf{TTM}$:

$$\mathfrak{G}(X_{v_q}) = \langle \lambda_{v_q}(-\mathcal{F}^\sharp(v_q)) + \tfrac{1}{2}(X_{v_q} - \mathsf{S}(v_q)), X_{v_q} - \mathsf{S}(v_q)\rangle_{\mathsf{T}_{v_q}\mathsf{TM}},$$

where S is the geodesic spray of $(\mathsf{M}, \mathfrak{g})$.

The following proposition is a corollary from Theorem 1.

Proposition 6 (Gibbs–Appell). *The GMA vector field $X_{\mathcal{C}}$ is the unique second-order vector field on $\mathcal{C}$ such that, on each fiber $\mathfrak{P}_{v_q}(\mathcal{C})$ of the holonomic prolongation of $\mathcal{C}$, $v_q \in \mathcal{C}$, minimizes the Gibbs–Appell function $\mathfrak{G}$. That is to say, for all $v_q \in \mathcal{C}$, we have*

$$\mathfrak{G}(X_{\mathcal{C}}(v_q)) = \min_{X_{v_q}\in\mathfrak{P}_{v_q}(\mathcal{C})} \mathfrak{G}(X_{v_q}).$$

3.2. Hölder's and Hertz's principles

In this subsection we show that, if the external force $\mathcal{F}$ derives from a potential $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$, the d'Alembert–Chetaev trajectories of $(\mathsf{M}, \mathrm{K}, \mathrm{V}, \mathcal{C})$ satisfy Hölder's principle—Theorem 2. In the *free mechanics* case (i.e. $\mathcal{F} = 0$), if the constraint is a cone, they also Hertz's principle of minimal geodesic curvature—Theorem 3.

3.2.1. Hölder's principle

We assume in this subsection that the external force $\mathcal{F}$ of the constrained mechanical system derives from a potential $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$, and we define the Lagrangian $\mathsf{L} : \mathsf{TM} \to \mathbb{R}$ by $\mathsf{L}(v_q) := \mathrm{K} - \mathrm{V} \circ \tau_{\mathsf{M}}$.

Hölder's principle—Theorem 2—will be stated in terms of some Banach manifolds which we describe below.

3.2.1.1. Some spaces of curves. Given $k \geq 0$ and a closed interval $[a, b] \subset \mathbb{R}$, we denote by $\mathsf{C}^k(\mathsf{M}, [a, b])$ the set of all curves $\gamma : [a, b] \to \mathsf{M}$ of class C^k. For $k \geq 1$, we denote by $\mathsf{H}^k(\mathsf{M}, [a, b])$ the set of all curves $\gamma : [a, b] \to \mathsf{M}$ of class H^k (a curve $\gamma : [a, b] \to \mathsf{M}$ is of class H^k if, taking a smooth embedding $\iota : \mathsf{M} \to \mathbb{R}^N$, whose existence is ensured by Whitney's theorem, $\iota \circ \gamma$ is a curve of class H^k in $\mathbb{R}^N$, that is to say, it is absolutely continuous and its derivative belongs to H^{k-1}, with $\mathsf{H}^0 = \mathsf{L}^2$; for $k \geq 1$, this definition does not depend on the choice of the embedding). If the interval $[a, b]$ is fixed and there is no risk of confusion, we use the abbreviated notations $\mathsf{C}^k(\mathsf{M})$ and $\mathsf{H}^k(\mathsf{M})$ instead of $\mathsf{C}^k(\mathsf{M}, [a, b])$ and $\mathsf{H}^k(\mathsf{M}, [a, b])$, respectively. These sets (for $k \geq 0$ in the C^k case, and $k \geq 1$ in the H^k case) admit Banach manifold structures (i.e. smooth manifolds modeled on Banach spaces, see [32,33] or [2], for example) naturally defined—see [17,21,42,44] or [16]. More precisely, the spaces H^k, $k \geq 1$, admit Hilbert manifold structures. Such smooth manifold structures are such that, given a proper smooth embedding $\iota : \mathsf{M} \to \mathbb{R}^N$ (which exists, by Whitney's theorem), then the application $(\iota\circ) : \gamma \mapsto i \circ \gamma$ is a smooth embedding of $\mathsf{C}^k(\mathsf{M})$ (respectively, $\mathsf{H}^k(\mathsf{M})$) into the Banach space $\mathsf{C}^k(\mathbb{R}^N)$ (respectively, into the Hilbert space $\mathsf{H}^k(\mathbb{R}^N)$) and $\mathsf{C}^k(\mathsf{M})$ is closed in $\mathsf{C}^k(\mathbb{R}^N)$ (respectively, $\mathsf{H}^k(\mathsf{M})$ is closed in $\mathsf{H}^k(\mathbb{R}^N)$). This property determines univocally the smooth manifold structures of $\mathsf{C}^k(\mathsf{M})$ and $\mathsf{H}^k(\mathsf{M})$. In particular, the manifolds $\mathsf{C}^k(\mathsf{M})$ and $\mathsf{H}^k(\mathsf{M})$ are metrizable (hence, paracompact) and separable. The inclusions $\mathsf{C}^k(\mathsf{M}) \to \mathsf{H}^k(\mathsf{M}) \to \mathsf{C}^{k-1}(\mathsf{M})$, $k \geq 1$, are smooth and have dense images. Besides, given a finite dimensional smooth manifold N and a smooth map $\phi : \mathsf{M} \to \mathsf{N}$, the map $(\phi\circ) : \gamma \mapsto \phi \circ \gamma$ is smooth from $\mathsf{C}^k(\mathsf{M})$ (respectively, $\mathsf{H}^k(\mathsf{M})$) with values in $\mathsf{C}^k(\mathsf{N})$ (respectively, $\mathsf{H}^k(\mathsf{N})$). For all $\gamma \in \mathsf{C}^k(\mathsf{M})$ (respectively, $\gamma \in \mathsf{H}^k(\mathsf{M})$), the tangent space at γ is the set of all sections of TM along γ of class C^k (respectively, of class H^k), that is to say:

$$\mathsf{T}_\gamma \mathsf{C}^k(\mathsf{M}) = \mathsf{C}^k(\gamma^*\mathsf{TM}) = \{X \in \mathsf{C}^k(\mathsf{TM}) | \tau_{\mathsf{M}} \circ X = \gamma\}$$

and similarly for $\mathsf{T}_\gamma \mathsf{H}^k(\mathsf{M})$. Hence, the tangent bundles $\tau_{\mathsf{C}^k(\mathsf{M})} : \mathsf{TC}^k(\mathsf{M}) \to \mathsf{C}^k(\mathsf{M})$ and $\tau_{\mathsf{H}^k(\mathsf{M})} : \mathsf{TH}^k(\mathsf{M}) \to \mathsf{H}^k(\mathsf{M})$ are naturally isomorphic to, respectively, $(\tau_{\mathsf{M}}\circ) : \mathsf{C}^k(\mathsf{TM}) \to \mathsf{C}^k(\mathsf{M})$ and $(\tau_{\mathsf{M}}\circ) : \mathsf{H}^k(\mathsf{TM}) \to \mathsf{H}^k(\mathsf{M})$.

More generally, given a smooth finite dimensional vector bundle $\pi_E : E \to \mathsf{M}$, we have smooth vector bundles $(\pi_E\circ) : \mathsf{C}^k(E) \to \mathsf{C}^k(\mathsf{M})$, for $k \geq 0$, and $(\pi_E\circ) : \mathsf{H}^k(E) \to$

$\mathsf{H}^k(\mathsf{M})$, for $k \geq 1$. These constructions are functorial, that is, given a smooth vector bundle morphism $\phi : E \to F$ over $\tilde{\phi} : \mathsf{M} \to \mathsf{N}$, we obtain smooth vector bundle morphisms $(\phi\circ) : \mathsf{C}^k(E) \to \mathsf{C}^k(F)$ over $(\tilde{\phi}\circ) : \mathsf{C}^k(\mathsf{M}) \to \mathsf{C}^k(\mathsf{N})$, for $k \geq 0$, and $(\phi\circ) : \mathsf{H}^k(E) \to \mathsf{H}^k(F)$ over $(\tilde{\phi}\circ) : \mathsf{H}^k(\mathsf{M}) \to \mathsf{H}^k(\mathsf{N})$, for $k \geq 1$.

3.2.1.2. The initial and the endpoint mappings. With the notation described above, let $k \geq 0$ and let us consider the map $ev_i : \mathsf{C}^k(\mathsf{M}) \to \mathsf{M}$ defined by $\gamma \mapsto \gamma(a)$, called the *initial point mapping*. This map is clearly smooth: taking a smooth embedding $\mathsf{M} \to \mathbb{R}^N$ given by Whitney's theorem, $ev_i : \mathsf{C}^k(\mathbb{R}^N) \to \mathbb{R}^N$ is linear continuous, hence its restriction to the embedded submanifold $\mathsf{C}^k(\mathsf{M})$ is smooth and takes values in the embedded submanifold $\mathsf{M} \subset \mathbb{R}^N$. Moreover, its tangent map at $\gamma \in \mathsf{C}^k(\mathsf{M})$ is given by $X \in \mathsf{T}_\gamma \mathsf{C}^k(\mathsf{M}) \equiv \mathsf{C}^k(\gamma^*\mathsf{TM}) \mapsto X(a) \in \mathsf{T}_{\gamma(a)}\mathsf{M}$, which is clearly surjective, and its kernel splits (i.e. admits a closed complementary subspace), since it has finite codimension in $\mathsf{T}_\gamma \mathsf{C}^k(\mathsf{M})$. Hence, we have shown that ev_i is a smooth submersion. Given $p \in \mathsf{M}$, its inverse image $ev_i^{-1}[p]$ is a closed embedded submanifold of $\mathsf{C}^k(\mathsf{M})$, which we denote henceforth by $\mathsf{C}^k(\mathsf{M}, p)$, and its tangent space at $\gamma \in \mathsf{C}^k(\mathsf{M}, p)$ is given by $\{X \in \mathsf{T}_\gamma \mathsf{C}^k(\mathsf{M}) | X(a) = 0\}$.

We can apply the same arguments we have used for the initial point mapping to conclude that the *endpoint mapping* $ev_f : \mathsf{C}^k(\mathsf{M}) \to \mathsf{M}$, $\gamma \mapsto \gamma(b)$, is also a smooth submersion. Given $p \in \mathsf{M}$, the restriction of ev_f to the embedded submanifold $\mathsf{C}^k(\mathsf{M}, p)$ is still a smooth submersion, by the same arguments; the inverse image by this last map of $q \in \mathsf{M}$ is a closed embedded submanifold of $\mathsf{C}^k(\mathsf{M}, p)$, which we denote by $\mathsf{C}^k(\mathsf{M}, p, q)$. The tangent space at $\gamma \in \mathsf{C}^k(\mathsf{M}, p, q)$ is given by $\{X \in \mathsf{T}_\gamma \mathsf{C}^k(\mathsf{M}) | X(a) = 0, X(b) = 0\}$.

All that we have done for the C^k case also applies to the Sobolev spaces H^k, for $k \geq 1$: we use the same notation for the initial and endpoint mappings, and we have corresponding closed embedded submanifolds $\mathsf{H}^k(\mathsf{M}, p)$, $\mathsf{H}^k(\mathsf{M}, p, q) \subset \mathsf{H}^k(\mathsf{M})$, given $p, q \in \mathsf{M}$.

3.2.1.3. Hölder's principle. In Theorem 2, we denote by $\mathsf{H}^1(C_{\dot{\gamma}}, [a, b], \gamma(a), \gamma(b))$ the closed linear subspace of $\mathsf{T}_\gamma \mathsf{H}^1(\mathsf{M}, [a, b])$ formed by the infinitesimal variations $\eta \in \mathsf{T}_\gamma \mathsf{H}^1(\mathsf{M}, [a, b], \gamma(a), \gamma(b))$ such that, for all $t \in [a, b]$, $\eta(t)$ is a virtual velocity at $\dot{\gamma}(t) \in \mathcal{C}$, i.e. $\mathsf{H}^1(C_{\dot{\gamma}}, [a, b], \gamma(a), \gamma(b)) := \{\eta \in \mathsf{T}_\gamma \mathsf{H}^1(\mathsf{M}, [a, b], \gamma(a), \gamma(b)) | \eta(t) \in C_{\dot{\gamma}(t)}$ a.e. on $[a, b]\}$.

We consider the *Lagrangian functional* $\mathcal{L} : \gamma \mapsto \int_a^b \mathrm{L}(\dot{\gamma})$ induced by L as a smooth map $\mathsf{H}^1(\mathsf{M}, [a, b]) \to \mathbb{R}$. The smoothness of $\mathcal{L}$ on $\mathsf{H}^1(\mathsf{M}, [a, b])$ follows from the fact that $\mathcal{L}$ is the difference of the smooth maps $\gamma \in \mathsf{H}^1(\mathsf{M}, [a, b]) \mapsto \int_a^b \langle\dot{\gamma}, \dot{\gamma}\rangle$ and $\gamma \in \mathsf{H}^1(\mathsf{M}, [a, b]) \mapsto \int_a^b \mathrm{V} \circ \gamma$. The second of these maps is smooth, since it can be written as the composition of smooth maps $\left(\int_a^b\right) \circ (\mathrm{V}\circ)$, where $(\mathrm{V}\circ) : \mathsf{H}^1(\mathsf{M}, [a, b]) \to \mathsf{H}^1(\mathbb{R}, [a, b])$ and $\left(\int_a^b\right) : \mathsf{H}^1(\mathbb{R}, [a, b]) \to \mathbb{R}$. On the other hand, to check the smoothness of $\gamma \in \mathsf{H}^1(\mathsf{M}, [a, b]) \mapsto \int_a^b \langle\dot{\gamma}, \dot{\gamma}\rangle$, take an isometric embedding $(\mathsf{M}, \mathfrak{g}) \to \mathbb{R}^N$, for sufficiently large N, which exists by Nash–Moser's theorem; then $\gamma \in \mathsf{H}^1(\mathbb{R}^N, [a, b]) \mapsto \|\dot{\gamma}\|_{\mathsf{L}^2}^2$ is obviously smooth, and so is its restriction to the embedded submanifold $\mathsf{H}^1(\mathsf{M}, [a, b])$.

Theorem 2 (Hölder's principle). *A horizontal curve* $\gamma \in \mathsf{H}^2(\mathsf{M}, [a, b])$ *is a d'Alembert–Chetaev trajectory of the constrained mechanical system* $(\mathsf{M}, \mathrm{K}, \mathrm{V}, \mathcal{C})$ *if, and only if,* $\mathrm{d}\mathcal{L}(\gamma)$ *annihilates the subspace* $\mathsf{H}^1(C_{\dot{\gamma}}, [a, b], \gamma(a), \gamma(b))$ *of* $\mathsf{T}_\gamma \mathsf{H}^1(\mathsf{M}, [a, b])$.

Remark 2. In [49] we have constructed Banach manifold structures on the spaces of horizontal curves $\mathsf{H}^k(\mathsf{M}, \mathcal{C}, [a, b]) := \{\gamma \in \mathsf{H}^k(\mathsf{M}, [a, b]) | \gamma \text{ is horizontal}\}$, for $k \geq 2$ (and also on $\mathsf{C}^k(\mathsf{M}, \mathcal{C}, [a, b]) := \{\gamma \in \mathsf{C}^k(\mathsf{M}, [a, b]) | \gamma \text{ is horizontal}\}$, for $k \geq 1$), and then we have defined the *variational trajectories* of the constrained mechanical system as critical points of the Lagrangian functional $\mathcal{L}$ on these Banach manifolds of horizontal curves. For linear constraints, its well known (see [5], for instance) that these trajectories coincide with the d'Alembert–Chetaev trajectories (which, in the linear case, coincide with the classical d'Alembertian trajectories of the constrained mechanical system, i.e. the trajectories defined by d'Alembert's principle) if, and only if, the constraint is integrable (i.e. they coincide only for *holonomic* constraints). We have generalized this condition for the general (non-linear) case in [49].

3.2.2. Hertz's principle

We recall that, given a curve γ on M, its *geodesic curvature* $\kappa_\gamma(t)$ at $t \in \operatorname{dom}\gamma$ is given by $\|\nabla_{s|s=0}\tilde{\gamma}\|$, where $\tilde{\gamma}$ is a reparametrization by arc length of γ with $\tilde{\gamma}(0) = \gamma(t)$ (see [48]).

The theorem that closes this section states that, in the case of a free mechanics (i.e. if the external force $\mathcal{F}$ is null), if the constraint manifold $\mathcal{C}$ is a *cone* (i.e. $v_q \in \mathcal{C}$ implies $(\forall t > 0) t v_q \in \mathcal{C}$) the d'Alembert–Chetaev trajectories of $(\mathsf{M}, \mathrm{K}, 0, \mathcal{C})$ satisfy Hertz's principle of least geodesic curvature. That is to say, except for reparametrizations, a horizontal curve γ is a d'Alembert–Chetaev trajectory of $(\mathsf{M}, \mathrm{K}, 0, \mathcal{C})$ if, and only if, for each t on its domain, its geodesic curvature at t is the greatest lower bound of the set of the geodesic curvatures at t of all horizontal curves defined on a neighborhood of t and with the same velocity at t, $\dot{\gamma}(t)$.

Theorem 3 (Hertz's principle of least curvature). *Assume that the constraint manifold $\mathcal{C}$ is a cone, and let γ be a horizontal curve on M. Then there exists a reparametrization of γ which is a d'Alembert–Chetaev trajectory of $(\mathsf{M}, \mathrm{K}, 0, \mathcal{C})$ if, and only if, for all $t \in \operatorname{dom}\gamma$:*

$$\kappa_\gamma(t) = \min\{\kappa_\alpha(0) | \alpha : (-\varepsilon, \varepsilon) \to \mathsf{M} \textit{ horizontal with } \dot{\alpha}(0) = \dot{\gamma}(t)\}.$$

3.3. Conservation of energy

It is a well known fact that, for a linearly constrained mechanical system $(\mathsf{M}, \mathrm{K}, \mathrm{V}, \mathcal{D})$ on which the external force derives from a potential $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$, the *mechanical energy* $E_{\mathcal{C}} := \mathrm{K}|_{\mathcal{C}} + \mathrm{V} \circ \pi_{\mathcal{C}}$ is a first integral of the flow of the GMA vector field $X_{\mathcal{D}}$—see [41]. It is then natural to inquire under which conditions the same occurs for a general (non-linear) constraint $\mathcal{C}$. We show in Proposition 7, that this is a characteristic of homogeneous constraints, in the sense of the following definition.

Definition 10. We say that a constraint $\mathcal{C} \subset \mathsf{TM}$ is *homogeneous* if the Liouville vector field $Z \in \mathfrak{D}^1(\mathsf{TM})$ (i.e. the vector field on TM defined by $v_q \in \mathsf{TM} \mapsto \lambda_{v_q} v_q \in \mathsf{TTM}$) is tangent to $\mathcal{C}$.

Example 3. If $\mathcal{C}$ is a *cone* (i.e. if $v_q \in \mathcal{C}$ implies $(\forall t > 0) t v_q \in \mathcal{C}$), then it is a homogeneous constraint. Linear constraints and also the constraint from Example 1(e) are cones.

Proposition 7. *Let* $(\mathsf{M}, \mathfrak{g})$ *be a Riemannian manifold,* $\mathrm{K} : \mathsf{TM} \to \mathbb{R}$ *the kinetic energy induced by* $\mathfrak{g}$ *and* $\mathcal{C} \subset \mathsf{TM}$ *a constraint. The following conditions are equivalent*:

(i) *for all potential* $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$, *the mechanical energy* $E_{\mathcal{C}} = \mathrm{K}|_{\mathcal{C}} + \mathrm{V} \circ \pi_{\mathcal{C}}$ *is a first integral of the flow of the GMA vector field* $X_{\mathcal{C}}^{\mathrm{V}}$;
(ii) $\mathcal{C}$ *is a homogeneous constraint*.

We note that the implication (ii) $\Rightarrow$ (i) was already known in formulations slightly different from ours—see [10,15,35,51].

As a corollary, we show that, for fixed $(\mathsf{M}, \mathrm{K}, \mathcal{C})$, with $\mathcal{C}$ *closed* in TM—as is the case of a constraint given by Proposition 2—then the GMA vector field of the constrained mechanical system $(\mathsf{M}, \mathrm{K}, \mathrm{V}, \mathcal{C})$ conserves the mechanical energy for all potentials $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$ if, and only if, $\mathcal{C}$ is a linear constraint.

Corollary 1. *With the same hypothesis, if* $\mathcal{C}$ *is closed in* TM, *both conditions in the statement of* Proposition 7 *are equivalent to* $\mathcal{C}$ *being a linear constraint*.

Finally, the following corollary follows from the previous corollary and from [12,50]. We recall that a Poisson bracket $\{\cdot, \cdot\}$ on $\mathcal{C}$ is an $\mathbb{R}$-bilinear anti-symmetric form on $\mathfrak{F}(\mathcal{C})$, satisfying both Jacobi's identity (i.e. turning $\mathfrak{F}(\mathcal{C})$ a Lie algebra over $\mathbb{R}$) and Leibniz's identity (i.e., $\{\cdot, \cdot\}$ is a derivation on the second factor).

Corollary 2. *With the same notation, the following conditions are equivalent*:

1. $(\mathcal{C}, \{\cdot, \cdot\})$ *is a Poisson manifold, closed in* TM, *and for all* $\phi \in \mathfrak{F}(\mathcal{C})$ *of the form* $\phi = K|_{\mathcal{C}} + \mathrm{V} \circ \pi_{\mathcal{C}}$, $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$, *the GMA vector field* $X_{\mathcal{C}}$ *of* $(\mathsf{M}, \mathrm{K}, \mathrm{V}, \mathcal{C})$ *coincides with the Hamiltonian vector field* $\xi_{\phi}^{\mathcal{C}}$ *induced by* ϕ., *i.e.* $\xi_{\phi}^{\mathcal{C}}[\psi] = \{\psi, \phi\}$, *for all* $\psi \in \mathfrak{F}(\mathcal{C})$;
2. $\mathcal{C}$ *a completely integrable smooth vector sub-bundle of* TM, *i.e. it is a holonomic constraint*.

See also [10], where an "almost-Poisson" bracket is constructed for systems with non-holonomic constraints.

3.4. The Jacobi–Carathéodory metric tensor

Given an unconstrained mechanical system $(\mathsf{M}, \mathrm{K}, \mathrm{V})$, with the external force deriving from a potential $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$, it is well known the "Jacobi–Carathéodory theorem": for $e > 0$ such that $\mathrm{V} < e$ on M, this theorem allows, through the introduction of a convenient metric tensor on M (the so-called Jacobi–Carathéodory metric tensor $\mathfrak{g}_e$, see Definition 11) reduce the study of the trajectories of $(\mathsf{M}, \mathrm{K}, \mathrm{V})$ with energy $\mathrm{K} + \mathrm{V} \circ \tau_{\mathsf{M}} = \text{const.} = e$ to the study of the geodesics of the Riemannian manifold $(\mathsf{M}, \mathfrak{g}_e)$ with energy 1—see [1].

We assume, throughout this subsection, that the constraint manifold $\mathcal{C}$ is a cone. In particular, $\mathcal{C}$ is homogeneous, i.e. the Liouville vector field $Z \in \mathfrak{D}^1(\mathsf{TM})$ is tangent to $\mathcal{C}$. It then follows from Proposition 7 that, for all potentials $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$, the mechanical energy $\mathrm{K}|_{\mathcal{C}} + \mathrm{V} \circ \pi_{\mathcal{C}}$ is conserved by the flow of the GMA vector field of $(\mathsf{M}, \mathrm{K}, \mathrm{V}, \mathcal{C})$. In this subsection we generalize the Jacobi–Carathéodory theorem for a constrained mechanical system of this type—see Theorem 4.

Definition 11. With the above notation, assume that there exists $e > 0$ such that, for all $q \in \mathsf{M}$, $\mathrm{V}(q)e$. We define the *Jacobi–Carathéodory metric tensor* on M by

$$\mathfrak{g}_e := (e - \mathrm{V})\mathfrak{g}. \tag{10}$$

Theorem 4. *With the above notation, let* $\gamma : [a, b] \to \mathsf{M}$ *be a smooth horizontal curve such that* $\mathrm{K}(\dot{\gamma}) + \mathrm{V} \circ \gamma = \text{const.} = e$ *and let* $\tilde{\gamma} : [0, L] \to \mathsf{M}$ *be the reparametrization by arc length of* γ *in the Jacobi–Carathéodory metric* $\mathfrak{g}_e$. *Denote by* K_e *the kinetic energy associated to* $\mathfrak{g}_e$. *Then* γ *is a d'Alembert–Chetaev trajectory of the constrained mechanical system* $(\mathsf{M}, \mathrm{K}, \mathrm{V}, \mathcal{C})$ *if, and only if,* $\tilde{\gamma}$ *is a d'Alembert–Chetaev trajectory of the free constrained mechanical system* $(\mathsf{M}, \mathrm{K}_e, 0, \mathcal{C})$.

As a corollary from this theorem and from Theorem 3, we obtain the following.

Corollary 3. *With the same notation, there exists a reparametrization of* γ *which is a d'Alembert–Chetaev trajectory of the constrained mechanical system* $(\mathsf{M}, \mathrm{K}, \mathrm{V}, \mathcal{C})$ *if, and only if, it minimizes the geodesic curvature in the Jacobi–Carathéodory metric, in the sense that, for all* $t \in \operatorname{dom}\gamma$:

$$\kappa_\gamma(t) = \min\{\kappa_\alpha(0) | \alpha : (-\varepsilon, \varepsilon) \to \mathsf{M} \textit{ horizontal with } \dot{\alpha}(0) = \dot{\gamma}(t)\},$$

where the geodesic curvatures κ *are taken with respect to the Jacobi–Carathéodory metric tensor*.

3.5. The Liouville's theorem for the Gibbs–Maggi–Appell vector field

In this subsection, we fix a Riemannian manifold $(\mathsf{M}, \mathfrak{g})$ and a constraint $\mathcal{C} \subset \mathsf{TM}$. We denote by K the kinetic energy induced by the metric tensor $\mathfrak{g}$. Our aim, in this section, is to generalize to the context of constrained mechanical systems the celebrated Liouville's theorem on the conservation of volume: for all potentials $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$, the flow of the GMA vector field X_V of the (unconstrained) mechanical system $(\mathsf{M}, \mathrm{K}, \mathrm{V})$ preserves the Liouville volume—i.e. the volume form on TM induced by the Sasaki metric tensor $\mathfrak{g}_{\mathsf{TM}}$ defined in Section 3.1.2.

Firstly, we define a metric tensor on $\mathcal{C}$ through a construction which generalizes that of the definition of the Sasaki metric tensor on TM.

Definition 12 (The Sasaki metric tensor on $\mathcal{C}$). The *Sasaki metric tensor* or, simply, the *Sasaki metric* on $\mathcal{C}$ is the unique metric tensor $\mathfrak{g}_\mathcal{C}$ on $\mathcal{C}$ such that, for all $v_q \in \mathcal{C}$, $\lambda^\mathcal{C}_{v_q}|_{C_{v_q}} : C_{v_q} \to \mathrm{Ver}_{v_q}(\mathcal{C})$ and $\mathrm{H}^\mathcal{C}_{v_q} : \mathsf{T}_q\mathsf{M} \to \mathrm{Hor}_{v_q}(\mathcal{C})$ are linear isometries.

Thus, endowing $\mathcal{C}$ with the metric tensor $\mathfrak{g}_\mathcal{C}$, we have $\mathrm{Hor}(\mathcal{C}) = \mathrm{Ver}(\mathcal{C})^\perp$ and, for all $v_q \in \mathcal{C}$, $X_{v_q}, Y_{v_q} \in \mathsf{T}_{v_q}\mathcal{C}$:

$$\begin{aligned}\mathfrak{g}_{\mathcal{C}}(X_{v_q}, Y_{v_q}) &= \langle P_H \cdot X_{v_q}, P_H \cdot Y_{v_q}\rangle_{\mathrm{Hor}^{\mathcal{C}}_{v_q}} + \langle P_V \cdot X_{v_q}, P_V \cdot Y_{v_q}\rangle_{\mathrm{Ver}^{\mathcal{C}}_{v_q}} \\ &= \langle \mathsf{T}\pi_{\mathcal{C}} \cdot X_{v_q}, \mathsf{T}\pi_{\mathcal{C}} \cdot Y_{v_q}\rangle + \langle \mathcal{P}(v_q) \cdot \kappa \cdot X_{v_q}, \mathcal{P}(v_q) \cdot \kappa \cdot Y_{v_q}\rangle.\end{aligned}$$

Note that, in the unconstrained case, i.e. if $\mathcal{C} = \mathsf{TM}$, the Sasaki metric tensor on $\mathcal{C}$ coincides with the one previously defined in Section 3.1.2.

The smooth map given by the following definition has an important role in the generalization of Liouville's theorem.

Definition 13. We denote by $A : \mathcal{C} \to \mathsf{L}(\mathsf{TM}, \mathsf{TM})$ the smooth map defined by, for all $v_q \in \mathcal{C}$, $A(v_q) := \kappa \circ P_{\mathcal{C}} \circ \mathrm{H}_{v_q} : \mathsf{T}_q\mathsf{M} \to \mathsf{T}_q\mathsf{M}$, where κ is the connector induced by the Levi–Civita connection of $(\mathsf{M}, \mathfrak{g})$.

Remark 3. For a linear constraint $\mathcal{D}$, a direct computation shows that the map A of the previous definition is given by, for all $v_q \in \mathcal{D}$, $A(v_q) = B_{\mathcal{D}}(v_q)$, where $B_{\mathcal{D}} : \mathsf{TM} \oplus_{\mathsf{M}} \mathcal{D} \to \mathcal{D}^{\perp}$ is the *total second fundamental form of* $(\mathsf{M}, \mathfrak{g}, \mathcal{D})$—see [31]—and $B_{\mathcal{D}}(v_q) = B_{\mathcal{D}}(\cdot, v_q) : \mathsf{T}_q\mathsf{M} \to \mathcal{D}^{\perp}_q$. In this sense, the map A of the previous definition plays the role, in the non-linearly constrained case, of the total second fundamental form.

Note that, for all $v_q \in \mathcal{C}$, we have $A(v_q) = -\kappa \circ P_W \circ \mathrm{H}_{v_q}$, hence $\mathrm{Im}\, A(v_q) \subset C^{\perp}_{v_q}$. This follows from the following facts: (1) $\kappa \circ \mathrm{H}_{v_q} = 0$ and (2) $P_{\mathcal{C}} + P_W = \mathrm{id}_{i^*_{\mathcal{C}}(\mathsf{TTM})}$. Besides, given $X_{v_q} \in \mathsf{T}_{v_q}\mathsf{TM}$, we have $X_{v_q} \in \mathsf{T}_{v_q}\mathcal{C}$ if, and only if:

$$\mathcal{P}^{\perp}(v_q) \cdot \kappa \cdot X_{v_q} = A(v_q) \cdot \mathsf{T}\tau_{\mathsf{M}} \cdot X_{v_q}. \tag{11}$$

Indeed, the last equation is clearly equivalent to $\kappa \cdot P_W \cdot X_{v_q} = 0 \Leftrightarrow P_W \cdot X_{v_q} = 0$.

In order to enunciate Theorem 5, we shall make use of the following notation.

Notation. Given $q \in \mathsf{M}$, $v_q \in \mathcal{C}_q$ and $w_q \in \mathsf{T}_q\mathsf{M}$, we denote by $\mathbb{F}^*\mathcal{P}(v_q) \cdot w_q$ the adjoint map of $\mathbb{F}\mathcal{P}(v_q) \cdot w_q : \mathsf{T}_q\mathsf{M} \to \mathsf{T}_q\mathsf{M}$ with respect to the metric tensor. This defines the map: $\mathbb{F}^*\mathcal{P} : \mathcal{C} \to \mathsf{L}(\mathsf{TM}, \mathsf{L}(\mathsf{TM}, \mathsf{TM})) \equiv \mathsf{L}(\mathsf{TM} \otimes \mathsf{TM}, \mathsf{TM})$.

The main result of this subsection is the following theorem.

Theorem 5. *The Lebesgue measure on* $\mathcal{C}$ *induced by the Sasaki metric* $\mathfrak{g}_{\mathcal{C}}$ *is preserved by the flow of the GMA vector field* $X^{\mathrm{V}}_{\mathcal{C}}$ *of the constrained mechanical system* $(\mathsf{M}, \mathrm{K}, \mathrm{V}, \mathcal{C})$, *for all potentials* $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$, *if, and only if, the two following conditions are fulfilled, for all* $v_q \in \mathcal{C}$:

(i) $\mathrm{tr}\, A(v_q) = 0$;
(ii) $\mathrm{tr}\, \mathbb{F}^*\mathcal{P}(v_q)|_{C_{v_q} \times C_{v_q}} = 0$.

The proof of this theorem is based on the computation of the Levi–Civita connection of $(\mathcal{C}, \mathfrak{g}_{\mathcal{C}})$ with respect to a convenient moving frame and of the divergence of the GMA vector field; as a by-product of these computations, we generalize a result from [46] concerning the geodesics of the Sasaki metric—see Proposition 8. Other secondary results are Corollary 4, which gives a necessary and sufficient condition for the integrability of the horizontal sub-bundle of $\mathsf{T}\mathcal{C}$, and Corollary 5, which states that, if conditions (i) and (ii) in Theorem 5 are fulfilled, then, for each $q \in \mathsf{M}$, $\mathcal{C}_q$ is a minimal surface of $(\mathcal{C}, \mathfrak{g}_{\mathcal{C}})$. That is to say, a necessary condition for the Lebesgue measure on $\mathcal{C}$ induced by the Sasaki metric to be preserved by the flow of the GMA vector field $X_{\mathcal{C}}^{\mathrm{V}}$, for all potentials $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$, is that $(\mathcal{C}, \mathfrak{g}_{\mathcal{C}})$ admit a regular foliation by minimal surfaces.

Example 4. Conditions (i) and (ii) of Theorem 5 are satisfied by the constraint from Example 1(e). Indeed, in this example we have: $\mathsf{M} = \mathbb{R}^4$, $(\forall x \in \mathsf{M})\mathcal{C}_x = \{(v_1, v_2, v_3, v_4) \in \mathbb{R}^4_x \setminus \{\mathbb{O}_x\} | \det\begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} = 0\}$. We use the following notation: given $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$, we put $|\vec{v} := (v_4, -v_3, -v_2, v_1)$ and $v' := (-v_2, v_1, -v_4, v_3)$. For all $v_x = (x, v) \in \mathcal{C}$, we have $C^{\perp}_{v_x} = [\vec{v}] \subset \mathsf{T}_x\mathsf{M} = \mathbb{R}^4_x$, $C_{v_x} = [\vec{v}]^{\perp}$, $W_{v_x} = \lambda_{v_x} C^{\perp}_{v_x} = [(0, \vec{v})_{v_x}] \subset \mathsf{T}_{v_x}\mathsf{TM} = (\mathbb{R}^4 \times \mathbb{R}^4)_{v_x}$, $\mathsf{T}_{v_x}\mathcal{C} = \{Y_{v_x} = (Y_1, Y_2)_{v_x} \in \mathsf{T}_{v_x}\mathsf{TM} | Y_2 \in C_{v_x} = [\overrightarrow{v_x}]^{\perp}\}$. Hence, for all $x \in \mathsf{M}$, $v_x, w_x \in \mathcal{C}_x$:

$$A(v_x) \cdot w_x = -\kappa \cdot P_W \cdot \mathrm{H}_{v_x} w_x = -\kappa \cdot P_W \cdot (w, 0)_{v_x} = 0,$$

so $A \equiv 0$, i.e. condition (i) is trivially fulfilled.

On the other hand, a direct computation shows that, for all $x \in \mathsf{M}$, $v_x \in \mathcal{C}_x$, $w_x, s_x \in C_{v_x}$:

$$\mathbb{F}^*\mathcal{P}(v_x) \cdot (w_x, s_x) = -\langle s_x, \overrightarrow{w_x} \rangle \frac{\overrightarrow{v_x}}{\|v_x\|^2}.$$

Therefore

$$\mathbb{F}^*\mathcal{P}(v_x) \cdot (w_x, w_x) = -\langle w_x, \overrightarrow{w_x} \rangle \frac{\overrightarrow{v_x}}{\|v_x\|^2} = -2 \det \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} \frac{\overrightarrow{v_x}}{\|v_x\|^2}.$$

Using the last formula and the orthonormal basis $((v/\|v\|), (v'/\|v\|), (\overrightarrow{v'}/\|v\|))$ of $C_{v_x} = [\overrightarrow{v_x}]^{\perp}$ to compute $\mathbb{F}^*\mathcal{P}(v_x)|_{C_{v_x} \times C_{v_x}}$, we conclude that condition (ii) holds, as asserted.

Remark 4.

(a) Note that, for a linear constraint $\mathcal{D}$, conditions (i) and (ii) from Theorem 5 are equivalent to the condition derived in [31] for the conservation of the local volume form defined there, i.e. to the condition $\operatorname{tr} B_{\mathcal{D}^{\perp}}(q)|_{\mathcal{D}^{\perp}_q \times \mathcal{D}^{\perp}_q} = 0$ for all $q \in \mathsf{M}$, where $B_{\mathcal{D}} : \mathsf{TM} \oplus_{\mathsf{M}} \mathcal{D} \to \mathcal{D}^{\perp}$ is the *total second fundamental form of* $(\mathsf{M}, \mathfrak{g}, \mathcal{D})$. Indeed, it can be easily checked that the above mentioned volume form coincides with the Riemannian volume induced by the Sasaki metric $\mathfrak{g}_{\mathcal{D}}$ on $\mathcal{D}$. To check the equivalency of the conditions, note that, for a linear constraint, $P_{\mathcal{D}} = \mathsf{T}\mathcal{P}_{\mathcal{D}} : i^*_{\mathcal{D}}(\mathsf{TTM}) \to \mathsf{T}\mathcal{D}$, where $\mathcal{P}_{\mathcal{D}} : \mathsf{TM} \to \mathcal{D}$ is the orthogonal projection, and, for all $v_q \in \mathcal{D}$, $C_{v_q} = \mathcal{D}_q$. Hence, $\mathcal{P} : v_q \in \mathcal{D} \mapsto (\mathcal{P}_{\mathcal{D}})_q \in \mathsf{L}(\mathsf{TM}, \mathsf{TM})$ is constant on the fibers of $\pi_{\mathcal{D}} : \mathcal{D} \to \mathsf{M}$. Thus, $\mathbb{F}\mathcal{P} = 0$ and condition (ii) from Theorem 5 is trivially fulfilled.

Besides, as we have pointed out in Remark 3, a direct computation shows that, for all $v_q \in \mathcal{C}$, $A(v_q) = B_{\mathcal{D}}(v_q)$, where $B_{\mathcal{D}}(v_q) = B_{\mathcal{D}}(\cdot, v_q) : \mathsf{T}_q\mathsf{M} \to \mathcal{D}^{\perp}_q$. Therefore, given a smooth orthonormal frame $(X_1, \ldots, X_n)$ on a neighborhood U of q in M, adapted to $\mathcal{D}_q$ (i.e. such that $(X_1(q), \ldots, X_r(q))$ is a basis of $\mathcal{D}_q$, where $r = \operatorname{rk} \mathcal{D}$), we have

$$\operatorname{tr} A(v_q) = \sum_{i=1}^{n} \langle X_i(q), B_{\mathcal{D}}(X_i(q), v_q)\rangle = \sum_{i=l+1}^{n} \langle X_i(q), B_{\mathcal{D}}(X_i(q), v_q)\rangle$$
$$= - \sum_{i=l+1}^{n} \langle B_{\mathcal{D}^\perp}(X_i(q), X_i(q)), v_q\rangle = -\langle \operatorname{tr} B_{\mathcal{D}^\perp}|_{\mathcal{D}_q^\perp \times \mathcal{D}_q^\perp}, v_q\rangle$$

what shows that condition (i) from Theorem 5 is equivalent to, for all $q \in \mathsf{M}$, $\operatorname{tr} B_{\mathcal{D}^\perp}(q)|_{\mathcal{D}_q^\perp \times \mathcal{D}_q^\perp} = 0$.

(b) Also with respect to the linearly constrained case, we refer the reader to [9], where a necessary and sufficient condition for the existence of an invariant measure for the dynamics of generalized Chaplygin systems was obtained.

Corollary 4. *The vector sub-bundle* $\mathrm{Hor}(\mathcal{C})$ *of* $\mathsf{T}\mathcal{C}$ *is involutive if, and only if, for all* $q \in \mathsf{M}$, $v_q \in \mathcal{C}_q$, $w_q, z_q \in \mathsf{T}_q\mathsf{M}$:

$$\mathcal{P}(v_q) \cdot \mathsf{R}(w_q, z_q) \cdot v_q = \mathcal{P}(v_q) \cdot \mathbb{P}A(v_q) \cdot (z_q, w_q) - \mathcal{P}(v_q) \cdot \mathbb{P}A(v_q) \cdot (w_q, z_q). \tag{12}$$

Remark 5. In the case of a linear constraint $\mathcal{D}$, we have, for all $v_q \in \mathcal{C}$, $A(v_q) = B_{\mathcal{D}}(v_q) := B_{\mathcal{D}}(\cdot, v_q) : \mathsf{T}_q\mathsf{M} \to \mathcal{D}_q^\perp$, where $B_{\mathcal{D}} : \mathsf{TM} \oplus_{\mathsf{M}} \mathcal{D} \to \mathcal{D}^\perp$ is the total second fundamental form of $\mathcal{D}$. Computing the parallel derivative $\mathbb{P}A$ and using Gauss' formula, we conclude that Eq. (12) is equivalent to $\mathsf{R}^{\mathcal{D}} \equiv 0$, where $\mathsf{R}^{\mathcal{D}}$ is the curvature tensor of the connection induced on $\mathcal{D}$ by the Levi–Civita connection of $(\mathsf{M}, \mathfrak{g})$ and by the orthogonal projection $\mathcal{P}_{\mathcal{D}} : \mathsf{TM} \to \mathcal{D}$. That is to say, we have reobtained the well known fact that the horizontal sub-bundle $\mathrm{Hor}(\mathcal{D})$ is involutive if, and only if, the connection $\nabla^{\mathcal{D}}$ is *flat*.

Corollary 5. *Suppose that the Lebesgue measure on* $\mathcal{C}$ *induced by the metric tensor* $\mathfrak{g}_{\mathcal{C}}$ *is preserved by the flow of the GMA vector field* $X_{\mathcal{C}}^{\mathrm{V}}$ *of the constrained mechanical system* $(\mathsf{M}, \mathrm{K}, \mathrm{V}, \mathcal{C})$, *for all potentials* $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$. *Then, for all* $q \in \mathsf{M}$ *such that* $\mathcal{C}_q \neq \emptyset$, $\mathcal{C}_q$ *is a minimal surface of* $(\mathcal{C}, \mathfrak{g}_{\mathcal{C}})$; *that is to say, the Riemannian manifold* $(\mathcal{C}, \mathfrak{g}_{\mathcal{C}})$ *admits a regular foliation by minimal surfaces*.

The following proposition, which closes this section, generalizes a result from [46].

Definition 14. We say that a constraint $\mathcal{C} \subset \mathsf{TM}$ is *totally geodesic* if the geodesic spray from $(\mathsf{M}, \mathfrak{g})$ is tangent to $\mathcal{C}$.

Proposition 8. *Let* γ *be a d'Alembert–Chetaev trajectory of the free constrained mechanical system* $(\mathsf{M}, \mathrm{K}, 0, \mathcal{C})$. *Then* $\dot\gamma$ *is a geodesic of* $(\mathcal{C}, \mathfrak{g}_{\mathcal{C}})$ *if, and only if,* γ *is a geodesic of* $(\mathsf{M}, \mathfrak{g})$. *Hence, the canonical lifts of the d'Alembert–Chetaev trajectories of* $(\mathsf{M}, \mathrm{K}, 0, \mathcal{C})$ *are geodesics of* $(\mathsf{M}, \mathfrak{g})$ *if, and only if,* $\mathcal{C}$ *is totally geodesic*.

Remark 6.

(a) A linear constraint $\mathcal{D}$ is totally geodesic, i.e. $\kappa \cdot P_W \cdot S(v_q) = -B_{\mathcal{D}}(v_q, v_q) = 0$ for all $v_q \in \mathcal{D}$ if, and only if, $B_{\mathcal{D}}|^s_{\mathcal{D}\oplus_{\mathsf{M}}\mathcal{D}} = 0$, that is to say, if the symmetric part of the restriction of $B_{\mathcal{D}}$ to $\mathcal{D} \oplus_{\mathsf{M}} \mathcal{D}$ is identically null.
(b) Putting $\mathcal{C} = \mathsf{TM}$, we reobtain the result from [46] which states that the canonical lifts of the geodesics of $(\mathsf{M}, \mathfrak{g})$ are geodesics of $(\mathsf{TM}, \mathfrak{g}_{\mathsf{TM}})$.

4. Proof of the main results

4.1. *Gauss' principle and the d'Alembert–Chetaev trajectories*

Proof of Theorem 1. Let $R^A : \mathcal{C} \to \mathsf{TM}$ be the admissible reaction field for $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$ defined by, for all $v_q \in \mathcal{C}$:

$$R^A(v_q) := -\kappa \cdot P_W(X_{\mathcal{F}}(v_q)), \tag{13}$$

where $X_{\mathcal{F}} : v_q \in \mathsf{TM} \mapsto \mathsf{S}(v_q) + \lambda_{v_q}(\mathcal{F}^\sharp(v_q)) \in \mathsf{T}_{v_q}\mathsf{TM}$ is the GMA vector field of $(\mathsf{M}, \mathrm{K}, \mathcal{F})$.

Note that R^A is smooth and $X_{\mathcal{C}}^{R^A} = X_{\mathcal{F}}|_{\mathcal{C}} - P_W \circ X_{\mathcal{F}}|_{\mathcal{C}} = X_{\mathcal{C}}$ is the GMA vector field of $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$, hence R^A is, indeed, an admissible reaction field.

Let $R \in \mathfrak{R}$ and $v_q \in \mathcal{C}$. Let us define $w_q := R(v_q) - R^A(v_q) \in \mathsf{T}_q\mathsf{M}$. Then $\lambda_{v_q} w_q \in \mathsf{T}_{v_q}\mathcal{C} \cap \mathrm{Ver}_{v_q}(\mathsf{TM}) = \mathrm{Ver}_{v_q}(\mathcal{C})$, since it is obviously vertical and $\mathcal{P}^\perp_{v_q} \cdot w_q = \mathcal{P}^\perp_{v_q} \cdot R(v_q) - \mathcal{P}^\perp_{v_q} \cdot R^A(v_q) = 0$, by Eq. (8). On the other hand, we have $\lambda_{v_q}(R^A(v_q)) \in W_{v_q} = \mathrm{Ver}_{v_q}(\mathcal{C})^\perp \subset \mathrm{Ver}_{v_q}(\mathsf{TM})$, thus $\langle \lambda_{v_q} w_q, \lambda_{v_q}(R^A(v_q))\rangle = 0$. It then follows that

$$\begin{aligned} \langle \lambda_{v_q}(R(v_q)), \lambda_{v_q}(R(v_q))\rangle &= \langle \lambda_{v_q}(R^A(v_q)), \lambda_{v_q}(R^A(v_q))\rangle + \langle \lambda_{v_q} w_q, \lambda_{v_q} w_q\rangle \\ &\quad + 2\underbrace{\langle \lambda_{v_q}(R^A(v_q)), \lambda_{v_q} w_q\rangle}_{=0} \\ &\geq \langle \lambda_{v_q}(R^A(v_q)), \lambda_{v_q}(R^A(v_q))\rangle \end{aligned}$$

and the equality holds if, and only if, $\lambda_{v_q} w_q = 0$, i.e. $w_q = R(v_q) - R^A(v_q) = 0$.

Since $\lambda_{v_q} : \mathsf{T}_q\mathsf{M} \to \mathrm{Ver}_{v_q}(\mathsf{TM})$ is a linear isometry, and since $v_q \in \mathcal{C}$ and $R \in \mathfrak{R}$ were arbitrarily taken, we have shown that, for all admissible reactions $R \in \mathfrak{R}$ and for all $v_q \in \mathcal{C}$, $\|R^A(v_q)\| \leq \|R(v_q)\|$, and that, if $R \in \mathfrak{R}$ satisfies $(\forall v_q \in \mathcal{C})\|R^A(v_q)\| = \|R(v_q)\|$, then $R = R^A$. □

Proof of Proposition 6. It is sufficient to note that

$$\mathfrak{G}(X_{v_q}) = \tfrac{1}{2}\|X_{v_q} - X_{\mathcal{F}}(v_q)\|^2_{\mathfrak{g}_{\mathsf{TM}}} - \tfrac{1}{2}\|\lambda_{v_q}\mathcal{F}^\sharp(v_q)\|^2_{\mathfrak{g}_{\mathsf{TM}}},$$

where $X_{\mathcal{F}}$ is the GMA vector field of the (unconstrained) mechanical system $(\mathsf{M}, \mathrm{K}, \mathcal{F})$, and apply Theorem 1, the formula for $X^R_{\mathcal{C}}$ given by Eq. (7) and Lemma 1. □

Lemma 1. *Let* $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$ *be a constrained mechanical system and* $\mathfrak{P}(\mathcal{C})$ *the holonomic prolongation of* $\mathcal{C}$ (see Definition 4). Then, for all $v_q \in \mathcal{C}$:

$$\mathfrak{P}_{v_q}(\mathcal{C}) = \{X_{\mathcal{C}}^R(v_q) | R \in \mathfrak{R}\},$$

where $\mathfrak{R}$ is the set of admissible reactions for $(\mathsf{M}, \mathrm{K}, \mathcal{F}, \mathcal{C})$ and $X_{\mathcal{C}}^R$ is given by Eq. (7).

Proof. Given $v_q \in \mathcal{C}$, the inclusion $\{X_{\mathcal{C}}^R(v_q) | R \in \mathfrak{R}\} \subset \mathfrak{P}_{v_q}(\mathcal{C})$ is clear, since $X_{\mathcal{C}}^R$ is a second-order vector field on $\mathcal{C}$, for all $R \in \mathfrak{R}$. To check the other inclusion, let $X_{v_q} \in \mathfrak{P}_{v_q}(\mathcal{C})$. Since $\mathfrak{P}(\mathcal{C}) \subset J^2(\mathsf{M})$, there exists a second-order vector field $\tilde{X} : \mathsf{TM} \to \mathsf{TTM}$ such that $\tilde{X}(v_q) = X_{v_q}$. Let $X : \mathcal{C} \to \mathsf{T}\mathcal{C}$ be the vector field defined by $X(w_q) = P_{\mathcal{C}} \cdot \tilde{X}(w_q)$, for all $w_q \in \mathcal{C}$. As $X(v_q) = P_{\mathcal{C}} \cdot X_{v_q} = X_{v_q}$ (since $X_{v_q} \in \mathsf{T}_{v_q}\mathcal{C}$), we achieve the demonstration once we show that there exists an admissible reaction $R \in \mathfrak{R}$ such that $X = X_{\mathcal{C}}^R$. As a matter of fact, define $R : w_q \in \mathcal{C} \mapsto \kappa \cdot X(w_q) - \mathcal{F}^\sharp(v_q) \in \mathsf{TM}$. Then, the fact of X being a second-order vector field on $\mathcal{C}$ implies that $\mathcal{P}_{w_q}^\perp \cdot \kappa \cdot X(w_q) + \kappa \cdot P_W \cdot \mathsf{S}(v_q) = 0$, for all $w_q \in \mathcal{C}$, hence R satisfies Eq. (8), i.e. $R \in \mathfrak{R}$ and $X = X_{\mathcal{C}}^R$, as asserted. □

4.2. Hölder's and Hertz's principles

Proof of Theorem 2. Given $\eta \in \mathsf{T}_\gamma \mathsf{H}^1(\mathsf{M}, [a, b])$, let $s \in (-\varepsilon, \varepsilon) \mapsto \gamma_s \in \mathsf{H}^1(\mathsf{M}, [a, b])$ such that $(T\gamma_s/\mathrm{d}s)|_{s=0} = \eta$. Then we have

$$\begin{aligned} \mathrm{d}\mathcal{L}(\gamma) \cdot \eta &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_a^b \mathrm{K}(\dot{\gamma}_s) - \mathrm{V}(\gamma_s) = \int_a^b \langle \nabla_t \eta, \dot{\gamma} \rangle - \langle \operatorname{grad} \mathrm{V}(\gamma), \eta \rangle \\ &\overset{\gamma \in \mathsf{H}^2}{=} \langle \eta, \dot{\gamma} \rangle|_a^b - \int_a^b \langle \nabla_t \dot{\gamma} + \operatorname{grad} \mathrm{V}(\gamma), \eta \rangle. \end{aligned}$$

Hence, for all $\eta \in \mathsf{H}^1(C_{\dot{\gamma}}, [a, b], \gamma(a), \gamma(b))$:

$$\mathrm{d}\mathcal{L}(\gamma) \cdot \eta = -\int_a^b \langle \nabla_t \dot{\gamma} + \operatorname{grad} \mathrm{V}(\gamma), \eta \rangle. \tag{14}$$

Assume that γ is a d'Alembert–Chetaev trajectory, i.e. a solution of Newton's equation (3) with reaction term R^A (where R^A is given by Theorem 1). Since $(\forall v_q \in \mathcal{C})\mathcal{P}(v_q) \cdot R^A(v_q) = 0$, it follows that, for all $t \in [a, b]$, $\mathcal{P}_{\dot{\gamma}} \cdot (\nabla_t \dot{\gamma} + \operatorname{grad} \mathrm{V}(\gamma)) = 0$, hence $\mathrm{d}\mathcal{L}(\gamma) \cdot \eta = 0$ for all $\eta \in \mathsf{H}^1(C_{\dot{\gamma}}, [a, b], \gamma(a), \gamma(b))$.

Reciprocally, assume that $\mathrm{d}\mathcal{L}(\gamma) \cdot \eta = 0$ for all $\eta \in \mathsf{H}^1(C_{\dot{\gamma}}, [a, b], \gamma(a), \gamma(b))$. Then it follows from (14) that $\mathcal{P}_{\dot{\gamma}} \cdot (\nabla_t \dot{\gamma} + \operatorname{grad} \mathrm{V}(\gamma)) = 0$ a.e. on $[a, b]$. On the other hand, as γ is a horizontal curve, we must have $\mathcal{P}_{\dot{\gamma}}^\perp \cdot \nabla_t \dot{\gamma} = -\mathcal{P}_{\dot{\gamma}}^\perp \cdot \operatorname{grad} \mathrm{V}(\gamma) + R^A(\dot{\gamma})$ a.e. on $[a, b]$. Therefore, summing the two last equations, we conclude that γ satisfies Newton's equation (3) with reaction term R^A a.e. on $[a, b]$, i.e. γ is a d'Alembert–Chetaev trajectory of $(\mathsf{M}, \mathrm{K}, \mathrm{V}, \mathcal{C})$. □

Proof of Theorem 3.

(i) It is enough to consider curves which are parametrized by arc length. This is a consequence of the following facts: (1) the geodesic curvature is independent of the parametrization; (2) since $\mathcal{C}$ is a cone, if a curve is horizontal, so is its arc length reparametrization; (3) if a curve is a d'Alembert–Chetaev trajectory of $(\mathsf{M}, \mathrm{K}, 0, \mathcal{C})$, so is its arc length reparametrization (this follows from the fact that, by Proposition 7, the kinetic energy K is constant along the d'Alembert–Chetaev trajectories of $(\mathsf{M}, \mathrm{K}, 0, \mathcal{C})$).

(ii) Assume that γ is a horizontal curve parametrized by arc length, and take $t \in \operatorname{dom}\gamma$. Let $v_q := \dot{\gamma}(t) \in \mathcal{C}$. For any horizontal curve $\alpha : (-\varepsilon, \varepsilon) \to \mathsf{M}$ parametrized by arc length such that $\dot{\alpha}(0) = v_q$, we have $\kappa_\alpha(0)^2 = \|\nabla_{t|t=0}\dot{\alpha}\|^2 = \|\mathcal{P}_{v_q} \cdot \nabla_{t|t=0}\dot{\alpha}\|^2 + \|R^A(v_q)\|^2$, since α horizontal implies $\mathcal{P}^{\perp}_{v_q} \cdot \nabla_{t|t=0}\dot{\alpha} = R^A(v_q)$.

(iii) If γ is a d'Alembert–Chetaev trajectory, we have $\mathcal{P}_{v_q} \cdot \nabla_t\dot{\gamma} = 0$, hence $\kappa_\gamma(t)^2 = \|R^A(v_q)\|^2 \leq \|\mathcal{P}_{v_q} \cdot \nabla_{t|t=0}\dot{\alpha}\|^2 + \|R^A(v_q)\|^2 = \kappa_\alpha(0)^2$, for all $\alpha : (-\varepsilon, \varepsilon) \to \mathsf{M}$ parametrized by arc length such that $\dot{\alpha}(0) = v_q$. It then follows $\kappa_\gamma(t) = \min\{\kappa_\alpha(0) | \alpha : (-\varepsilon, \varepsilon) \to \mathsf{M}$ horizontal with $\dot{\alpha}(0) = \dot{\gamma}(t)\}$.

(iv) Reciprocally, assume that $\kappa_\gamma(t) = \min\{\kappa_\alpha(0) | \alpha : (-\varepsilon, \varepsilon) \to \mathsf{M}$ horizontal with $\dot{\alpha}(0) = \dot{\gamma}(t)\}$. Let $\alpha : (-\varepsilon, \varepsilon) \to \mathsf{M}$ be a d'Alembert–Chetaev trajectory with $\alpha(0) = q$ and $\dot{\alpha}(0) = v_q$. Then $\kappa_\gamma(t)^2 = \|R^A(v_q)\|^2 + \|\mathcal{P}_{\dot{\gamma}(t)} \cdot \nabla_t\dot{\gamma}\|^2 \leq \kappa_\alpha(0)^2 = \|R^A(v_q)\|^2$, hence $\mathcal{P}_{\dot{\gamma}(t)} \cdot \nabla_t\dot{\gamma} = 0$. Since $t \in \operatorname{dom}\gamma$ was arbitrarily taken, we conclude that γ is a d'Alembert–Chetaev trajectory. □

4.3. *Conservation of energy*

Proof of Proposition 7. Given $v_q \in \mathcal{C}$, we have $Z(v_q) \in \mathsf{T}_{v_q}\mathcal{C} \Leftrightarrow \lambda_{v_q} v_q \in \mathrm{Ver}_{v_q}(\mathcal{C}) \Leftrightarrow v_q = \kappa \cdot \lambda_{v_q} v_q \in C_{v_q}$. By the arbitrariness of $v_q \in \mathcal{C}$, we conclude that Z is tangent to $\mathcal{C}$ if, and only if, the following condition holds

(ii′) $(\forall v_q \in \mathcal{C}) v_q \in C_{v_q}$.

Assume that condition (ii′) holds, i.e. the constraint is homogeneous. Given $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$, let $\gamma : I \to \mathsf{M}$ be a d'Alembert–Chetaev trajectory of $(\mathsf{M}, \mathrm{K}, \mathrm{V}, \mathcal{C})$ defined on the interval $I \subset \mathbb{R}$. Then, for all $t \in I$, $\dot{\gamma} \in \mathcal{C}$, hence $\dot{\gamma} \in C_{\dot{\gamma}}$ by (ii′). Besides, as $\nabla_t\dot{\gamma} + \operatorname{grad}\mathrm{V}(\gamma) = R^A_{\mathrm{V}}(\dot{\gamma}) \in C^{\perp}_{\dot{\gamma}}$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\{\mathrm{K}(\dot{\gamma}) + \mathrm{V}(\gamma)\} &= \langle \nabla_t\dot{\gamma}, \dot{\gamma}\rangle + \langle \operatorname{grad}\mathrm{V}(\gamma), \dot{\gamma}\rangle = \langle \nabla_t\dot{\gamma} + \operatorname{grad}\mathrm{V}(\gamma), \dot{\gamma}\rangle \\ &= \langle R^A_{\mathrm{V}}(\dot{\gamma}), \dot{\gamma}\rangle = 0,\end{aligned} \tag{15}$$

so $\mathrm{K} + \mathrm{V} \circ \tau_{\mathsf{M}}$ is constant along $\dot{\gamma}$. Since γ was arbitrarily taken, it follows that $\mathrm{K} + \mathrm{V} \circ \tau_{\mathsf{M}}$ is a first integral of the GMA vector field $X^{\mathrm{V}}_{\mathcal{C}}$, for all $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$, i.e. condition (i) holds.

Reciprocally, assume that condition (ii′) is false, i.e. there exists $v_q \in \mathcal{C}$ such that $v_q \notin C_{v_q}$. Then, defining $v^{\perp}_q := \mathcal{P}^{\perp}_{v_q} v_q \in C^{\perp}_{v_q}$, we have $v^{\perp}_q \neq 0$. Take $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$ such that $\operatorname{grad}\mathrm{V}(q) = v^{\perp}_q + \kappa \cdot P_W \cdot \mathsf{S}(v_q)$; therefore, from Eq. (13) it follows that $R^A_{\mathrm{V}}(v_q) = -\kappa \cdot P_W \cdot X_{\mathrm{V}}(v_q) = v^{\perp}_q$.

Thus, by the same computation done in (15):

$$X_{\mathrm{V}}^{A}(v_q)[\mathrm{K}+\mathrm{V}\circ\tau_{\mathsf{M}}] = \langle R_{\mathrm{V}}^{A}(v_q), v_q\rangle = \langle v_q^{\perp}, v_q^{\perp}\rangle > 0$$

what shows that condition (i) does not hold. □

Proof of Corollary 1. Indeed, if $\mathcal{C}$ is a homogeneous constraint, it follows from Lemma 2, that, for each $q \in \mathsf{M}$, $\mathcal{C}_q$ is a linear subspace of $\mathsf{T}_q\mathsf{M}$. Moreover, since $\pi_{\mathcal{C}} : \mathcal{C} \to \mathsf{M}$ is a submersion and $\mathcal{C}_q = \pi_{\mathcal{C}}^{-1}[q]$, all subspaces $\mathcal{C}_q$, $q \in \mathsf{M}$, have the same dimension. We contend that $q \mapsto \mathcal{C}_q$ is a smooth distribution on M (i.e. it is locally generated by smooth sections).

As a matter of fact, let $q \in \mathsf{M}$ and $(e_1, \dots, e_k)$ be a basis of $\mathcal{C}_q$. As $\pi_{\mathcal{C}}$ is a submersion, there exist local smooth sections $X_1, \dots, X_k$ of $\pi_{\mathcal{C}} : \mathcal{C} \to \mathsf{M}$, defined on an open set $U \subset \mathsf{M}$ with $q \in U$, such that $X_i(q) = e_i$, for $1 \le i \le k$. By continuity, there exists an open neighborhood $\tilde{U} \subset U$ of q such that $\{X_1, \dots, X_k\}$ is linearly independent on $\tilde{U}$. Therefore, $\mathcal{C}|_{\tilde{U}}$ is generated by the smooth sections $X_1, \dots, X_k$. Finally, since $q \in \mathsf{M}$ was arbitrarily taken, we conclude that $\mathcal{C}$ is locally generated by smooth sections, i.e. $\mathcal{C}$ is a smooth distribution, as asserted. □

Lemma 2. *If $\mathcal{C} \subset \mathsf{TM}$ is a closed homogeneous constraint, then, for all $q \in \mathsf{M}$, $\mathcal{C}_q$ is a linear subspace of $\mathsf{T}_q\mathsf{M}$.*

Proof. Let $q \in \mathsf{M}$. We have

1. $\mathcal{C}_q$ is a closed embedded sub-manifold of $\mathsf{T}_q\mathsf{M}$.

   Indeed, we have already proven in Section 2.2 that $\mathcal{C}_q$ is an embedded sub-manifold of $\mathsf{T}_q\mathsf{M}$. The hypothesis of $\mathcal{C}$ being closed in TM implies that $\mathcal{C}_q$ is closed in TM, hence in $\mathsf{T}_q\mathsf{M}$.
2. For each $v_q \in \mathcal{C}_q$ and for each $t \ge 0$, we have $tv_q \in \mathcal{C}_q$.

   Indeed, for each $v_q \in \mathcal{C}$, the fact of Z being tangent to $\mathcal{C}$ implies that there exists $\varepsilon(v_q) > 0$ such that $\mathrm{e}^t v_q \in \mathcal{C}$ for $t \in (-\varepsilon(v_q), \varepsilon(v_q))$. Let $T_{v_q} := \sup\{t \in \mathbb{R} | \mathrm{e}^t v_q \in \mathcal{C}\}$ and $t_{v_q} := \inf\{t \in \mathbb{R} | \mathrm{e}^t v_q \in \mathcal{C}\}$. If $T_{v_q} < +\infty$, the fact of $\mathcal{C}$ being closed TM implies that $\mathrm{e}^{T_{v_q}} v_q \in \mathcal{C}$, hence there exists $t > T_{v_q}$ such that $\mathrm{e}^t v_q \in \mathcal{C}$, what is a contradiction; thus, $T_{v_q} = +\infty$. Similarly, $t_{v_q} = -\infty$. This shows that $\mathrm{e}^t v_q \in \mathcal{C}$ for all $t \in \mathbb{R}$, i.e. $tv_q \in \mathcal{C}$ for all $t > 0$. Again by the fact of $\mathcal{C}$ being closed in TM, it follows that $\mathbb{O}_q = 0v_q \in \mathcal{C}$, what concludes the proof of the assertion.
3. Identifying $T_{\mathbb{O}_q}(\mathcal{C}_q)$ with a linear subspace of $\mathsf{T}_q\mathsf{M}$, we assert that $\mathcal{C}_q = T_{\mathbb{O}_q}(\mathcal{C}_q)$, and this concludes the proof. As a matter of fact, let $w_q \in \mathcal{C}_q$ and define:

   $$\gamma : [0, +\infty) \to \mathcal{C}_q, \qquad t \mapsto tw_q.$$

   Then γ is a differentiable curve in $\mathcal{C}_q$ (at 0, this means that it is differentiable from the right), since it is differentiable as a curve with values in $\mathsf{T}_q\mathsf{M}$ and $\mathcal{C}_q$ is an embedded sub-manifold of $\mathsf{T}_q\mathsf{M}$, as we have seen. Thus

   $$w_q = \left.\frac{T\gamma}{\mathrm{d}t}\right|_{t=0} \in \mathsf{T}_{\mathbb{O}_q}(\mathcal{C}_q).$$

Since $w_q \in \mathcal{C}_q$ was arbitrarily taken, this shows that $\mathcal{C}_q \subset T_{\mathbb{O}_q}(\mathcal{C}_q)$. But $\mathcal{C}_q$ and $\mathsf{T}_{\mathbb{O}_q}(\mathcal{C}_q)$ are both embedded sub-manifolds of $\mathsf{T}_q\mathsf{M}$ with the same dimension; thus, if $\mathcal{C}_q \subset T_{\mathbb{O}_q}(\mathcal{C}_q)$, $\mathcal{C}_q$ must be an open sub-manifold of $T_{\mathbb{O}_q}(\mathcal{C}_q)$. As $\mathcal{C}_q$ is closed in $\mathsf{T}_q\mathsf{M}$, it must be also closed in $T_{\mathbb{O}_q}(\mathcal{C}_q)$, which is connected, since it is a vector space. Then $\mathcal{C}_q = T_{\mathbb{O}_q}(\mathcal{C}_q)$, as asserted. □

4.4. The Jacobi–Carathéodory metric tensor

Proof of Theorem 4.

(i) Let $g : [a, b] \to [0, L]$ be defined by $t \mapsto \int_a^t \sqrt{\mathfrak{g}_e(\dot\gamma, \dot\gamma)}$. Then $(\circ g) : \mathsf{H}^1(\mathsf{M}, [0, L]) \to \mathsf{H}^1(\mathsf{M}, [a, b])$ is a smooth diffeomorphism, and $(\circ g^{-1}) = (\circ h)$ (where $h : g^{-1} : [0, L] \to [a, b]$). We assert that the tangent map $\mathsf{T}_\gamma(\circ h)$ maps the linear subspace $\mathsf{H}^1(C_{\dot\gamma}, [a, b], \gamma(a), \gamma(b))$ isomorphically onto $\mathsf{H}^1(C_{\tilde\gamma'}, [0, L], \gamma(a), \gamma(b))$. Indeed, given $\eta \in \mathsf{T}_\gamma\mathsf{H}^1(\mathsf{M}, [a, b], \gamma(a), \gamma(b))$, by definition we have $\eta \in \mathsf{H}^1(C_{\dot\gamma}, [a, b], \gamma(a), \gamma(b))$ if, and only if, $\eta(t) \in C_{\dot\gamma(t)}$ a.e. on $[a, b]$. Hence, $\eta \in \mathsf{H}^1(C_{\dot\gamma}, [a, b], \gamma(a), \gamma(b))$ if, and only if, $\tilde\eta := \mathsf{T}(\circ h) \cdot \eta = \eta \circ h$ satisfies $\tilde\eta(s) = \eta \circ h(s) \in C_{\dot\gamma(h(s))}$ a.e. on $[0, L]$. Since, for all $s \in [0, L]$, $\tilde\gamma'(s) = \dot\gamma(h(s))h'(s)$ and $h'(s) > 0$, the assertion will be proved once we show that, for all $v_q \in \mathcal{C}$ and for all $t > 0$, $C_{v_q} = C_{tv_q} \subset \mathsf{T}_q\mathsf{M}$. As a matter of fact, given $t > 0$, the hypothesis of $\mathcal{C}$ being a cone ensures that

$$\mu^t : \mathcal{C} \to \mathcal{C}, \qquad v_q \mapsto tv_q$$

is a well defined smooth diffeomorphism. Besides, it is clear that μ^t preserves fibers, i.e. for all $q \in \mathsf{M}$, $\mu^t(\mathcal{C}_q) = \mathcal{C}_q$. Therefore, for all $v_q \in \mathcal{C}$, we have $\mathsf{T}\mu^t \cdot \mathsf{T}_{v_q}(\mathcal{C}_q) = \mathsf{T}_{tv_q}(\mathcal{C}_q)$ and, applying the connector κ_{TM} to both members of this last equation, we conclude that $C_{v_q} = C_{tv_q}$, as asserted.

(ii) Let $\mathcal{L}_e : \mathsf{H}^1(\mathsf{M}, [0, L]) \to \mathbb{R}$ be the Lagrangian functional induced by K_e, i.e. defined by $\gamma \mapsto \int_a^b \mathrm{K}_e(\dot\gamma)$. Using the fact that $\mathrm{K} \circ \dot\gamma + \mathrm{V} \circ \gamma = \text{const.} = e$, a direct computation shows that, for all $J \in \mathsf{T}_\gamma\mathsf{H}^1(\mathsf{M}, [a, b])$:

$$\mathrm{d}\mathcal{L}(\gamma) \cdot J = \sqrt{2}\, \mathrm{d}\mathcal{L}_e(\tilde\gamma) \cdot \tilde J, \tag{16}$$

where $\tilde J = \mathsf{T}_\gamma(\circ h) \cdot J = J \circ h \in \mathsf{T}_{\tilde\gamma}\mathsf{H}^1(\mathsf{M}, [0, L])$. Since $\mathsf{T}_\gamma(\circ h)$ maps $\mathsf{H}^1(C_{\dot\gamma}, [a, b], \gamma(a), \gamma(b))$ isomorphically onto $\mathsf{H}^1(C_{\tilde\gamma'}, [0, L], \gamma(a), \gamma(b))$, Eq. (16) shows that $\mathrm{d}\mathcal{L}(\gamma) \cdot \mathsf{H}^1(C_{\dot\gamma}, [a, b], \gamma(a), \gamma(b)) = \{\mathbb{O}\}$ if, and only if, $\mathrm{d}\mathcal{L}_e(\tilde\gamma) \cdot \mathsf{H}^1(C_{\tilde\gamma'}, [0, L], \gamma(a), \gamma(b)) = \{\mathbb{O}\}$. The proof then follows from Theorem 2. □

4.5. The Liouville's theorem for the Gibbs–Maggi–Appell vector field

In order to demonstrate Theorem 5, we define a convenient moving frame on $\mathcal{C}$ and we compute the Levi–Civita connection $\nabla^{\mathcal{C}}$ of $(\mathcal{C}, \mathfrak{g}_{\mathcal{C}})$ and the divergence of the GMA vector field $X_{\mathcal{C}}^{\mathrm{V}}$ in terms of this moving frame. This will be done in the following definitions and lemmata.

Definition 15. Given $v_q \in \mathcal{C}$, let $(X_1, \dots, X_n)$ be an orthonormal frame field of $(\mathsf{M}, \mathfrak{g})$ defined on an open neighborhood U of $q \in \mathsf{M}$. Let us define, for $1 \leq i \leq n$ and for all $w_q \in \mathcal{C}_U := \pi_{\mathcal{C}}^{-1}[U]$:

$$X_i^H(w_q) := \mathrm{H}_{w_q}^{\mathcal{C}}(X_i(q)), \qquad X_i^V(w_q) := \lambda_{w_q}^{\mathcal{C}}(X_i(q)).$$

We can assume that $(\mathcal{P}(v_q)\cdot X_1(q), \dots, \mathcal{P}(v_q)\cdot X_l(q))$ is a basis of C_{v_q}, where $l = \mathrm{rk}\,\mathrm{Ver}(\mathcal{C})$. Then, taking vertical lifts, we conclude that $(X_1^V(v_q), \dots, X_l^V(v_q))$ is a basis of $\mathrm{Ver}_{v_q}(\mathcal{C})$. By continuity, $(X_1^V, \dots, X_l^V)$ forms a frame field on the vector bundle $\mathrm{Ver}(\mathcal{C})$ on a neighborhood $\mathcal{U}$ of v_q on $\mathcal{C}$.

Thus, we have constructed a frame field $\mathsf{F} = (X_1^{H_\mathcal{C}}, \dots, X_n^{H_\mathcal{C}}, X_1^{V_\mathcal{C}}, \dots, X_l^{V_\mathcal{C}})$ of $\mathcal{C}$ on a neighborhood $\mathcal{U}$ of v_q.

Note that this frame field is not orthonormal, except for its "horizontal" part, i.e. $\langle X_i^H, X_j^H\rangle = \delta_{ij}$, for $1 \leq i, j \leq n$. Note also that, if $\mathcal{C} = \mathsf{TM}$, we have $l = n$ and the frame field is orthonormal, and we can take $\mathcal{U} = \tau_{\mathsf{M}}^{-1}[U]$.

Notation. For the sake of clearness, we use indices i, j, k for horizontal vectors r, s, u for vertical vectors.

Proof of Corollary 4. Corollary 4 is a direct consequence of the next lemma. □

Lemma 3. *Using the notation from* Definition 15, *we have, for* $1 \leq i, j, r, s \leq n$:

$$\begin{aligned}
[X_i^H, X_j^H](v_q) &= \mathrm{H}_{v_q}^{\mathcal{C}}([X_i, X_j](q)) + \lambda_{v_q}^{\mathcal{C}}\{\mathcal{P}(v_q)\cdot \mathsf{R}(X_j(q), X_i(q))\cdot v_q \\
&\quad + \mathcal{P}(v_q)\cdot \mathbb{P}A(v_q)\cdot (X_j(q), X_i(q)) - \mathcal{P}(v_q)\cdot \mathbb{P}A(v_q)\cdot (X_i(q), X_j(q))\}, \\
[X_r^V, X_s^V](v_q) &= \lambda_{v_q}^{\mathcal{C}}\{\mathcal{P}(v_q)\cdot \mathbb{F}\mathcal{P}(v_q)\cdot (X_r(q), X_s(q)) \\
&\quad - \mathcal{P}(v_q)\cdot \mathbb{F}\mathcal{P}(v_q)\cdot (X_s(q), X_r(q))\}, \\
[X_i^H, X_r^V](v_q) &= \lambda_{v_q}^{\mathcal{C}}\{\mathcal{P}(v_q)\cdot \nabla_{X_i(q)} X_r + \mathcal{P}(v_q)\cdot \mathbb{P}\mathcal{P}(v_q)\cdot (X_i(q), X_r(q)) \\
&\quad - \mathcal{P}(v_q)\cdot \mathbb{F}A(v_q)\cdot (X_r(q), X_i(q))\},
\end{aligned} \tag{17}$$

where R *is the curvature tensor of* $(\mathsf{M}, \mathfrak{g})$.

Proof. We demonstrate only the first formula, since the technique used the compute the others is the same. Note that, since, for $1 \leq i, j \leq n$, X_i^H is $\pi_{\mathcal{C}}$-related to X_i and X_i^V is $\pi_{\mathcal{C}}$-related to zero, we immediately obtain $\mathsf{T}\pi_{\mathcal{C}}\cdot [X_i^H, X_j^H](v_q) = [X_i, X_j](q)$, $\mathsf{T}\pi_{\mathcal{C}}\cdot [X_i^V, X_j^V](v_q) = 0$ and $\mathsf{T}\pi_{\mathcal{C}}\cdot [X_i^H, X_j^V](v_q) = 0$.

We have, for $1 \leq i, j, r, s \leq n$ and for all $f \in \mathfrak{F}(\mathsf{TM})$:

(1)
$$\begin{aligned}
X_i^H[f](v_q) &= \mathbb{F}f(v_q)\cdot \kappa \cdot X_i^H(v_q) + \mathbb{P}f(v_q)\cdot \mathsf{T}\pi_{\mathcal{C}}\cdot X_i^H(v_q) \\
&= \mathbb{F}f(v_q)\cdot A(v_q)\cdot X_i(q) + \mathbb{P}f(v_q)\cdot X_i(q)
\end{aligned}$$

and

$$\begin{aligned}
X_r^V[f](v_q) &= \mathbb{F}f(v_q)\cdot \kappa \cdot X_r^V(v_q) + \mathbb{P}f(v_q)\cdot \mathsf{T}\pi_{\mathcal{C}}\cdot X_r^V(v_q) \\
&= \mathbb{F}f(v_q)\cdot \mathcal{P}(v_q)\cdot X_r(q).
\end{aligned}$$

(2) Let $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{C}$ be a curve on $\mathcal{C}$ such that $(T\gamma/\mathrm{d}t)|_{t=0} = X_i^H(v_q)$, and let $q(t) := \pi_{\mathcal{C}} \circ \gamma(t)$. We have

$$
\begin{aligned}
X_i^H[X_j^H[f]](v_q) = {} & \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (X_j^H[f] \circ \gamma) \\
= {} & \nabla^{\mathbb{R}_\mathsf{M}}_{t|t=0}\{\mathbb{F} f(\gamma(t)) \cdot A(\gamma(t)) \cdot X_j(q(t)) + \mathbb{P} f(\gamma(t)) \cdot X_j(q(t))\} \\
= {} & \mathbb{F}^2 f(v_q) \cdot (\kappa \cdot X_i^H(v_q), \kappa \cdot X_j^H(v_q)) \\
& + \mathbb{P}\mathbb{F} f(v_q) \cdot (\mathsf{T}\pi_{\mathcal{C}} \cdot X_i^H(v_q), \kappa \cdot X_j^H(v_q)) \\
& + \mathbb{F}\mathbb{P} f(v_q) \cdot (\kappa \cdot X_i^H(v_q), \mathsf{T}\pi_{\mathcal{C}} \cdot X_j^H(v_q)) \\
& + \mathbb{P}^2 f(v_q) \cdot (\mathsf{T}\pi_{\mathcal{C}} \cdot X_i^H(v_q), \mathsf{T}\pi_{\mathcal{C}} \cdot X_j^H(v_q)) \\
& + \mathbb{F} f(v_q) \cdot \{\mathbb{P}A(v_q) \cdot (X_i(q), X_j(q)) + A(v_q) \cdot \nabla_{X_i(q)} X_j\} \\
& + \mathbb{P} f(v_q) \cdot \nabla_{X_i(q)} X_j.
\end{aligned}
$$

Hence, by Proposition 1, it follows from the last equation that

$$
\begin{aligned}
[X_i^H, X_j^H][f](v_q) = {} & \mathbb{F} f(v_q) \cdot \{\mathsf{R}(X_j(q), X_i(q)) \cdot v_q + \mathbb{P}A(v_q) \cdot (X_i(q), X_j(q)) \\
& - \mathbb{P}A(v_q) \cdot (X_j(q), X_i(q)) + A(v_q) \cdot [X_i, X_j](q)\} \\
& + \mathbb{P} f(v_q) \cdot [X_i, X_j](q)
\end{aligned}
$$

and, since $f \in \mathfrak{F}(\mathsf{TM})$ was arbitrarily taken, we conclude that

$$
\mathsf{T}\pi_{\mathcal{C}} \cdot [X_i^H, X_j^H](v_q) = [X_i, X_j](q)
$$

and

$$
\begin{aligned}
& \mathcal{P}(v_q) \cdot \kappa \cdot [X_i^H, X_j^H](v_q) \\
& \quad = \mathcal{P}(v_q) \cdot \mathsf{R}(X_j(q), X_i(q)) \cdot v_q + \mathcal{P}(v_q) \cdot \mathbb{P}A(v_q) \cdot (X_i(q), X_j(q)) \\
& \qquad - \mathcal{P}(v_q) \cdot \mathbb{P}A(v_q) \cdot (X_j(q), X_i(q)).
\end{aligned}
$$

Finally, writing $[X_i^H, X_j^H](v_q) = \mathrm{H}^{\mathcal{C}}_{v_q} \cdot \mathsf{T}\pi_{\mathcal{C}} \cdot [X_i^H, X_j^H](v_q) + \lambda^{\mathcal{C}}_{v_q} \cdot \mathcal{P}(v_q) \cdot \kappa \cdot [X_i^H, X_j^H]$ (v_q), we obtain the asserted formula for $[X_i^H, X_j^H](v_q)$. □

Lemma 4. *Denoting by* $\nabla^{\mathcal{C}}$ *the Levi–Civita connection of* $(\mathcal{C}, \mathfrak{g}_{\mathcal{C}})$, *and using the notation from* Definition 15, *we have, for* $1 \le i, j, r, s \le n$:

$$
\begin{aligned}
\nabla^{\mathcal{C}}_{X_i^H(v_q)} X_j^H = {} & \mathrm{H}^{\mathcal{C}}_{v_q}(\nabla_{X_i(q)} X_j) + \tfrac{1}{2}\lambda^{\mathcal{C}}_{v_q}\{\mathcal{P}(v_q) \cdot \mathsf{R}(X_j(q), X_i(q)) \cdot v_q \\
& - \mathcal{P}(v_q) \cdot \mathbb{P}A(v_q) \cdot (X_j(q), X_i(q)) + \mathcal{P}(v_q) \cdot \mathbb{P}A(v_q) \cdot (X_i(q), X_j(q))\}, \\
\nabla^{\mathcal{C}}_{X_r^V(v_q)} X_s^V = {} & \tfrac{1}{2}\mathrm{H}^{\mathcal{C}}_{v_q} \cdot A^*(v_q) \cdot \{\mathbb{F}^*\mathcal{P}(v_q) \cdot (X_r(q), X_s(q)) + \mathbb{F}^*\mathcal{P}(v_q) \cdot (X_s(q), X_r(q))\} \\
& + \lambda^{\mathcal{C}}_{v_q}\{\mathcal{P}(v_q) \cdot \mathbb{F}\mathcal{P}(v_q) \cdot (X_r(q), X_s(q))\}
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
\langle \nabla^{\mathcal{C}}_{X_i^H(v_q)} X_r^V, X_j^H(v_q)\rangle &= \tfrac{1}{2}\langle \mathcal{P}(v_q)\cdot X_r(q), -\mathcal{P}(v_q)\cdot \mathsf{R}(X_j(q), X_i(q))\cdot v_q - \mathcal{P}(v_q) \\
&\quad \cdot \mathbb{P}A(v_q)\cdot (X_i(q), X_j(q)) + \mathcal{P}(v_q)\cdot \mathbb{P}A(v_q)\cdot (X_j(q), X_i(q))\rangle, \\
\langle \nabla^{\mathcal{C}}_{X_i^H(v_q)} X_r^V, X_s^V(v_q)\rangle &= \langle \mathcal{P}(v_q)\cdot X_s(q), \mathbb{F}\mathcal{P}(v_q)\cdot (X_i(q), X_r(q)) + \nabla_{X_i(q)} X_r\rangle \\
&\quad - \tfrac{1}{2}\langle \mathcal{P}(v_q)\cdot X_s(q), \mathbb{F}A(v_q)\cdot (X_r(q), X_i(q))\rangle \\
&\quad + \tfrac{1}{2}\langle \mathcal{P}(v_q)\cdot X_r(q), \mathbb{F}A(v_q)\cdot (X_s(q), X_i(q))\rangle.
\end{aligned} \tag{19}
$$

Proof. It is a consequence from Lemma 3 and from Koszul's formula:

$$
\begin{aligned}
2\langle \nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle \\
&\quad + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle. \qquad \square
\end{aligned}
$$

Definition 16. Using the notation from Definition 15, let $(U, (\theta^1, \dots, \theta^n))$ the dual coframe field of $(U, (X_1, \dots, X_n))$. For $1 \le i \le n$, let $\hat{\theta}^i : \mathsf{T}U \to \mathbb{R}$ be defined by, for all $w_q \in \mathsf{T}U$:

$$\hat{\theta}^i(w_q) := \theta^i(q)\cdot w_q.$$

Let $\mathcal{S} := P_{\mathcal{C}} \circ \mathsf{S}|_{\mathcal{C}} : \mathcal{C} \to \mathsf{T}\mathcal{C}$, where S is the geodesic spray of $(\mathsf{M}, \mathfrak{g})$, and $\mathcal{V} : \mathcal{C} \to \mathsf{T}\mathcal{C}$ be defined by, for all $v_q \in \mathcal{C}$, $\mathcal{V}(v_q) := P_{\mathcal{C}}\cdot \lambda_{v_q}(-\mathrm{grad}\,\mathrm{V}(q)) = \lambda^{\mathcal{C}}_{v_q}(-\mathrm{grad}\,\mathrm{V}(q))$.

We have, for all $v_q \in \mathcal{U}$, $\mathcal{S}(v_q) = P_{\mathcal{C}}\mathrm{H}_{v_q}(v_q) = \mathrm{H}^{\mathcal{C}}_{v_q}(v_q) = \sum_{j=1}^n \hat{\theta}^j(v_q) X_j^H(v_q)$ and $\mathcal{V}(v_q) = \lambda^{\mathcal{C}}_{v_q}(-\mathrm{grad}\,\mathrm{V}(q)) = -\sum_{i=1}^n \hat{\theta}^i(\mathrm{grad}\,\mathrm{V}(q)) X_i^V(v_q)$.

That is to say, $X^{\mathrm{V}}_{\mathcal{C}} = \mathcal{S} + \mathcal{V}$, where

$$\mathcal{S}|_{\mathcal{U}} = \sum_{j=1}^n \hat{\theta}^j X_j^H, \qquad \mathcal{V}|_{\mathcal{U}} = -\sum_{i=1}^n (\hat{\theta}^i \circ \mathrm{grad}\,\mathrm{V} \circ \pi_{\mathcal{C}}|_{\mathcal{U}}) X_i^V. \tag{20}$$

4.5.1. Proof of Liouville's theorem

We can now demonstrate Theorem 5. The demonstration is a direct consequence of the following proposition, which gives the expression of the divergence of the GMA vector field, with respect to the Riemannian volume.

Proposition 9. *For all $v_q \in \mathcal{C}$, $\mathrm{div}\, X^{\mathrm{V}}_{\mathcal{C}}$ is given by the following formula*:

$$\mathrm{div}\, X^{\mathrm{V}}_{\mathcal{C}}(v_q) = \mathrm{tr}\, A(v_q) + \langle \mathrm{tr}\, \mathbb{F}^*\mathcal{P}(v_q)|_{C_{v_q}\times C_{v_q}}, R^A_{\mathrm{V}}(v_q)\rangle. \tag{21}$$

Proof. We can assume that, at the point $q \in \mathsf{M}$, the orthonormal frame field $(U, (X_1, \dots, X_n))$ is adapted to C_{v_q}, i.e. $(X_1(q), \dots, X_l(q))$ is an orthonormal basis of C_{v_q} and $(X_{l+1}(q), \dots, X_n(q))$ is an orthonormal basis of $C^{\perp}_{v_q}$. Then, we have

$$\mathrm{div}\, X^{\mathrm{V}}_{\mathcal{C}}(v_q) = \sum_{i=1}^n \langle X_i^H(v_q), \nabla^{\mathcal{C}}_{X_i^H(v_q)} X^{\mathrm{V}}_{\mathcal{C}}\rangle + \sum_{r=1}^l \langle X_r^V(v_q), \nabla^{\mathcal{C}}_{X_r^V(v_q)} X^{\mathrm{V}}_{\mathcal{C}}\rangle. \tag{22}$$

We now use Eq. (20) and Lemma 4 to compute the terms on the second member of (22). □

Lemma 5. *For all $q \in \mathsf{M}$, $v_q \in \mathcal{C}_q$, $w_q \in \mathsf{T}_q\mathsf{M}$:*

(i) $\mathbb{F}\mathcal{P}(v_q) = -\mathbb{F}\mathcal{P}^\perp(v_q)$ *and* $\mathbb{P}\mathcal{P}(v_q) = -\mathbb{P}\mathcal{P}^\perp(v_q)$;
(ii) $\mathcal{P}(v_q) \circ \{\mathbb{F}\mathcal{P}(v_q) \cdot w_q\} = \{\mathbb{F}\mathcal{P}(v_q) \cdot w_q\} \circ \mathcal{P}^\perp(v_q)$ *and* $\mathcal{P}(v_q) \circ \{\mathbb{P}\mathcal{P}(v_q) \cdot w_q\} = \{\mathbb{P}\mathcal{P}(v_q) \cdot w_q\} \circ \mathcal{P}^\perp(v_q)$;
(iii) $\mathcal{P}(v_q) \circ \{\mathbb{F}A(v_q) \cdot w_q\} = -\{\mathbb{F}\mathcal{P}(v_q) \cdot w_q\} \circ A_{v_q}$ *and* $\mathcal{P}(v_q) \circ \{\mathbb{P}A(v_q) \cdot w_q\} = -\{\mathbb{P}\mathcal{P}(v_q) \cdot w_q\} \circ A_{v_q}$.

Proof. The three assertions follow, respectively, by derivation of the identities $(\forall v_q \in \mathcal{C})\mathcal{P}(v_q) + \mathcal{P}^\perp(v_q) = \mathrm{id}_{\mathsf{T}_q\mathsf{M}}$, $\mathcal{P}(v_q) \circ \mathcal{P}^\perp(v_q) = 0$ and $\mathcal{P}(v_q) \circ A(v_q) = 0$. □

Proof of Theorem 5. By Eq. (21), it is clear that $\operatorname{div} X_{\mathcal{C}}^{\mathrm{V}}$ is identically null on $\mathcal{C}$, for all potentials $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$, if conditions (i) and (ii) are satisfied. Reciprocally, assume that $\operatorname{div} X_{\mathcal{C}}^{\mathrm{V}}$ is identically null on $\mathcal{C}$ for all $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$. Let us fix $v_q \in \mathcal{C}$. As $\{\operatorname{grad} \mathrm{V}(q) | \mathrm{V} \in \mathfrak{F}(\mathsf{M})\} = \mathsf{T}_q\mathsf{M}$, and since $\mathcal{P}^\perp(v_q) : \mathsf{T}_q\mathsf{M} \to \mathsf{T}_q\mathsf{M}$ is onto $C_{v_q}^\perp$, there exists $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$ such that $\mathcal{P}^\perp(v_q) \cdot \operatorname{grad} \mathrm{V}(q) = \kappa \cdot P_W \cdot \mathsf{S}(v_q)$, i.e. such that $R_{\mathrm{V}}^A(v_q) = 0$. Thus, for this V, we conclude from Eq. (21) that $\operatorname{div} X_{\mathcal{C}}^{\mathrm{V}}(v_q) = 0$ implies $\operatorname{tr} A(v_q) = 0$. It then follows that

$$\operatorname{div} X_{\mathcal{C}}^{\mathrm{V}}(v_q) = \langle \operatorname{tr} \mathbb{F}^*\mathcal{P}(v_q)|_{C_{v_q} \times C_{v_q}}, R_{\mathrm{V}}^A(v_q) \rangle \tag{23}$$

and this expression must be zero for all $\mathrm{V} \in \mathfrak{F}(\mathsf{M})$. Again by the fact that $\{\operatorname{grad} \mathrm{V}(q) | \mathrm{V} \in \mathfrak{F}(\mathsf{M})\} = \mathsf{T}_q\mathsf{M}$ and that $\mathcal{P}^\perp(v_q) : \mathsf{T}_q\mathsf{M} \to \mathsf{T}_q\mathsf{M}$ is onto $C_{v_q}^\perp$, we conclude that $\{R_{\mathrm{V}}^A(v_q) | \mathrm{V} \in \mathfrak{F}(\mathsf{M})\} = C_{v_q}^\perp$. Hence, it follows from (23) that $\mathcal{P}^\perp(v_q) \cdot \operatorname{tr} \mathbb{F}^*\mathcal{P}(v_q) = 0$. We contend that $\mathcal{P}^\perp(v_q) \cdot \operatorname{tr} \mathbb{F}^*\mathcal{P}(v_q) = \operatorname{tr} \mathbb{F}^*\mathcal{P}(v_q)$; as $v_q \in \mathcal{C}$ was arbitrarily taken, this will achieve the demonstration, since conditions (i) and (ii) will be verified. Indeed, for all $q \in \mathsf{M}$, $v_q \in \mathcal{C}_q$, $w_q, z_q, s_q \in \mathcal{C}_{v_q}$, we have

$$\begin{aligned} \langle \mathbb{F}^*\mathcal{P}(v_q) \cdot (w_q, z_q), s_q \rangle &= \langle z_q, \mathbb{F}\mathcal{P}(v_q) \cdot (w_q, s_q) \rangle \\ &\overset{z_q \in C_{v_q}}{=} \langle z_q, \mathcal{P}(v_q)\mathbb{F}\mathcal{P}(v_q) \cdot (w_q, s_q) \rangle \\ &\overset{\text{Lemma 5}}{=} \langle z_q, \mathbb{F}\mathcal{P}(v_q) \cdot (w_q, \mathcal{P}^\perp(v_q) \cdot s_q) \rangle \overset{s_q \in C_{v_q}}{=} 0, \end{aligned}$$

what shows that, for all $v_q \in \mathcal{C}$, $\mathbb{F}^*\mathcal{P}(v_q)|_{C_{v_q} \times C_{v_q}} : C_{v_q} \times C_{v_q} \to C_{v_q}^\perp$. □

Proof of Corollary 5. Indeed, let $q \in \mathsf{M}$ such that $\mathcal{C}_q \neq \emptyset$ and let $B_{\mathcal{C}_q}$ be the second fundamental form of $\mathcal{C}_q$. Given $v_q \in \mathcal{C}_q$ and $X_{v_q}, Y_{v_q} \in \mathsf{T}_{v_q}\mathcal{C}_q$, we assert that

$$\begin{aligned} B_{\mathcal{C}_q}(X_{v_q}, Y_{v_q}) = \tfrac{1}{2}\mathrm{H}_{v_q}^{\mathcal{C}} \cdot A^*(v_q) \cdot \{&\mathbb{F}^*\mathcal{P}(v_q) \cdot (\kappa \cdot X_{v_q}, \kappa \cdot Y_{v_q}) \\ &+ \mathbb{F}^*\mathcal{P}(v_q) \cdot (\kappa \cdot Y_{v_q}, \kappa \cdot X_{v_q})\}. \end{aligned}$$

As a matter of fact, using the notation from Definition 15, it follows from (18) that, for $1 \le r, s \le n$:

$$\begin{aligned} B_{\mathcal{C}_q}(X_r^V(v_q), X_s^V(v_q)) &= P_H \cdot \nabla^{\mathcal{C}}_{X_r^V(v_q)} X_s^V \\ &= \tfrac{1}{2}\mathrm{H}^{\mathcal{C}}_{v_q} \cdot A^*(v_q) \cdot \{\mathbb{F}^*\mathcal{P}(v_q) \cdot (X_r(q), X_s(q)) \\ &\quad + \mathbb{F}^*\mathcal{P}(v_q) \cdot (X_s(q), X_r(q))\}. \end{aligned}$$

Therefore, for all $v_q \in \mathcal{C}_q$, $\operatorname{tr} B_{\mathcal{C}_q}(v_q) = \mathrm{H}^{\mathcal{C}}_{v_q} \cdot A^*(v_q) \cdot \operatorname{tr} \mathbb{F}^*\mathcal{P}(v_q)|_{C_{v_q} \times C_{v_q}}$, hence $\operatorname{tr} B_{\mathcal{C}_q}(v_q) = 0$ if $\operatorname{tr} \mathbb{F}^*\mathcal{P}(v_q)|_{C_{v_q} \times C_{v_q}} = 0$. □

Proof of Proposition 8. Let γ be a d'Alembert–Chetaev trajectory of $(\mathsf{M}, \mathrm{K}, 0, \mathcal{C})$, i.e. for all $t \in \operatorname{dom}\gamma$, we have

$$\ddot{\gamma}(t) = P_{\mathcal{C}} \cdot \mathsf{S}(\dot{\gamma}(t)) = \mathcal{S}(\dot{\gamma}(t)).$$

Hence

$$\nabla^{\mathcal{C}}_t \ddot{\gamma} = \nabla^{\mathcal{C}}_t (\mathcal{S} \circ \dot{\gamma}).$$

Let us fix $t \in \operatorname{dom}\gamma$ and let $p := \gamma(t) \in \mathsf{M}$, $w_p := \dot{\gamma}(t) \in \mathcal{C}$. Let $\mathsf{F} = (X_1^H, \ldots, X_n^H, X_1^V, \ldots, X_l^V)$ be a frame field on $\mathcal{C}$ on an open neighborhood $\mathcal{U}$ of w_p in $\mathcal{C}$, like in Definition 15. As usual, we can assume that, on the point $p \in \mathsf{M}$, $(X_1(p), \ldots, X_n(p))$ is an orthonormal frame adapted to C_{w_p}, so that $(X_1^V(w_p), \ldots, X_l^V(w_p))$ is an orthonormal basis of $\mathrm{Ver}^{\mathcal{C}}_{w_p}$. Let $(U, (\theta^1, \ldots, \theta^n))$ be the dual coframe of $(U, (X_1, \ldots, X_n))$, as in Definition 16. We have

$$\mathcal{S}|_{\mathcal{U}} = \sum_{j=1}^{n} \hat{\theta}^j X_j^H. \tag{24}$$

Therefore

$$\nabla^{\mathcal{C}}_t \ddot{\gamma} = \sum_{i=1}^{n} \hat{\theta}^i(w_p) \nabla^{\mathcal{C}}_{X_i^H(w_p)} \mathcal{S}. \tag{25}$$

Using Lemma 4 and Eq. (24) to compute the second member of (25), we conclude that $\nabla^{\mathcal{C}}_t \ddot{\gamma} = 0$ if, and only if, $\langle \kappa \cdot P_W \cdot \mathsf{S}(w_p), X_k(p)\rangle = 0$ for $1 \le k \le n$, i.e. if, and only if, $\kappa \cdot P_W \cdot \mathsf{S}(w_p) = 0$, what is equivalent to $P_W \cdot \mathsf{S}(w_p) = 0$. Since $t \in \operatorname{dom}\gamma$ was arbitrarily taken, we have shown that $\dot{\gamma}$ is a geodesic of $(\mathcal{C}, \mathfrak{g}_{\mathcal{C}})$ if, and only if, $P_W \cdot \mathsf{S}(\dot{\gamma}(t)) = 0$ for all $t \in \operatorname{dom}\gamma$. As γ is a d'Alembert–Chetaev trajectory, this is equivalent to $\ddot{\gamma}(t) = P_{\mathcal{C}} \cdot \mathsf{S}(\dot{\gamma}(t)) = \mathsf{S}(\dot{\gamma}(t))$ for all $t \in \operatorname{dom}\gamma$. Thus, $\dot{\gamma}$ is a geodesic of $(\mathcal{C}, \mathfrak{g}_{\mathcal{C}})$ if, and only if, γ is a geodesic of $(\mathsf{M}, \mathfrak{g})$, as asserted. □

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